THE HOMOGENEOUS SOBOLEV METRIC OF ORDER ONE ON DIFFEOMORPHISM GROUPS ON THE REAL LINE

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Abstract. In this article we study Sobolev metrics of order one on diffeomorphism groups on the real line. We prove that the space $\operatorname{Diff}_1(\mathbb{R})$ equipped with the homogenous Sobolev metric of order one is a flat space in the sense of Riemannian geometry, as it is isometric to an open subset of a mapping space equipped with the flat L^2-metric. Here $\operatorname{Diff}_1(\mathbb{R})$ denotes the extension of the group of all either compactly supported, rapidly decreasing or H^∞-diffeomorphisms, that allows for a shift towards infinity. In particular this result provides an analytic solution formula for the corresponding geodesic equation, the non-periodic Hunter-Saxton equation. In addition we show that one can obtain a similar result for the two-component Hunter-Saxton equation and discuss the case of the non-homogenous Sobolev one metric which is related to the Camassa-Holm equation.

1. Introduction

Over the last decades it has been shown that various prominent PDEs arise as geodesic equations on certain infinite dimensional manifolds. This phenomenon has been first observed in the groundbreaking paper [1] by Arnold for the incompressible Euler equation, which is the geodesic equation on the group of volume preserving diffeomorphisms with respect to the right invariant L^2-metric. It was shown that this geometric approach can be extended to a whole variety of other PDEs used in hydrodynamics: the Camassa-Holm equation [6, 17], the Constantin-Lax-Majda equation [40, 3] or the Korteweg-de Vries equation [39, 4] to name but a few examples.

It was later realized that the geometric interpretation could be used to prove results about the behaviour of the PDEs. The first such result was by Ebin and Marsden [9], where they showed the local well-posedness of Euler equations. Similar techniques were then applied to other PDEs, that arise as geodesic equations, see e.g. [7, 8, 10, 13, 5].

The analysis in this paper is mainly concerned with the Hunter-Saxton (HS) equation. For the periodic case it was shown in [16] that the HS-equation is the geodesic equation on the homogenous space $\operatorname{Diff}(S^1)/S^1$ with respect to the homogenous Sobolev metric of order one or $\dot{H}^1$-metric. Lenells used this geometric

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interpretation in [26, 27] to construct an analytic solution formula for the equation. In fact he showed that the Riemannian manifold $\big(\mathrm{Diff}(S^1)/S^1, \dot H^1\big)$ is isometric to an open subset of a L^2-sphere in the space $C^\infty(S^1,\mathbb{R})$ of periodic functions and therefore obtained an explicit formula for the corresponding geodesics on $\mathrm{Diff}(S^1)/S^1$.

The aim of this article is twofold. First we extend the results of [26] to groups of real analytic and ultra-differentiable diffeomorphisms and show that the solutions of the HS-equation are analytic in time and space. Second we consider the $\dot H^1$-metric and the HS-equation on the real line. Our main result can be paraphrased as follows (see Section 4).

Theorem. *The non-periodic Hunter-Saxton equation is the geodesic equation on* $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *with respect to the right invariant* $\dot H^1$*-metric. Furthermore, the space* $\big(\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot H^1\big)$ *is isometric to an open subset in* $\big(\mathcal{A}(\mathbb{R}), L^2\big)$ *and is thus a flat space in the sense of Riemannian geometry.*

Here $\mathcal{A}(\mathbb{R})$ denotes one of the function spaces $C_c^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$ or $H^\infty(\mathbb{R})$ and $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ is an extension over the diffeomorphism group including shifts near $+\infty$; see Section 2.

The first surprising fact is that the normal subgroups $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ do not admit the geodesic equation (or the Levi-Civita covariant derivative) for this right invariant metric. The extended group $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ admits it but in a weaker sense than realized by [1] and follow-up papers; see 3.2. We also sketch Arnold's curvature formula in this weaker setting.

The second surprising fact is that $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ with the $\dot H^1$-metric is a flat Riemannian manifold as opposed to $\mathrm{Diff}(S^1)$, which is a positively curved one.

The main ingredient for the proof of this result is the R-map, which allows us to isometrically embed $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ with the right invariant $\dot H^1$-metric as an open subset of a (flat) pre-Hilbert space. This phenomenon has also been observed for the space $\mathrm{Imm}(S^1,\mathbb{R}^2)/S^1$ of plane curves modulo parameter translations (see. e.g., [25, 42, 2]), and also on the semi-direct product space $\mathrm{Diff}(S^1) \ltimes C^\infty(S^1,\mathbb{R})$ (the corresponding geodesic equation is the periodic two-component HS-equation; see [28]).

In the periodic case we extend the results to groups of real analytic diffeomorphisms and ultra-differentiable diffeomorphisms of certain types and show that the HS-equation has solutions that are real analytic or ultra-differentiable if the initial diffeomorphism is.

In the Section 5 we apply the same techniques to treat the two-component HS-equation on the real line. We discuss the existence of the geodesic equation and construct an isometry between the configuration space $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}) \ltimes \mathcal{A}(\mathbb{R})$ and an open subset of a pre-Hilbert space.

Finally we generalize the constructions to the case of the right-invariant non-homogenous H^1-metric on $\mathrm{Diff}(S^1)$, whose geodesic equation is the dispersion-free Camassa-Holm equation. In this case we define an R-map, whose image is a subspace of a pre-Hilbert space, not open any more.

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2. Some diffeomorphism groups on the real line and the circle

The group of all orientation preserving diffeomorphisms $\mathrm{Diff}(\mathbb{R})$ is not an open subset of $C^\infty(\mathbb{R},\mathbb{R})$ endowed with the compact C^∞-topology and so it is not a smooth manifold with charts in the usual sense. One option is to consider it as a Lie group in the cartesian closed category of Frölicher spaces, see [20, Section 23] with the structure induced by the injection $f \mapsto (f, f^{-1}) \in C^\infty(\mathbb{R},\mathbb{R}) \times C^\infty(\mathbb{R},\mathbb{R})$. Alternatively one can use the theory of smooth manifolds based on smooth curves instead of charts from [35], [36], which agrees with the usual theory up to Banach manifolds. We will in this paper restrict our attention to subgroups of the whole diffeomorphism group, which are smooth Fréchet manifolds.

Let us first briefly recall the definition of a regular Lie groups in the sense of [21], see also [20, 38.4]. A smooth Lie group G with Lie algebra $\mathfrak{g} = T_eG$ is called regular if the following holds:

- For each smooth curve $X \in C^\infty(\mathbb{R},\mathfrak{g})$ there exists a smooth curve $g \in C^\infty(\mathbb{R},G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) &= e \\ \partial_t g(t) &= T_e(\mu^{g(t)})X(t) = X(t).g(t)\ . \end{cases} \tag{1}$$

  The curve g, if it exists, is uniquely determined by its initial value $g(0)$.
- The map $\mathrm{evol}^r_G(X) = g(1)$ where g is the unique solution of (1), considered as a map $\mathrm{evol}^r_G : C^\infty(\mathbb{R},\mathfrak{g}) \to G$ is C^∞-smooth.

2.1. **The group** $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R})$**.** The 'largest' regular Lie group in $\mathrm{Diff}(\mathbb{R})$ with charts is the group of all diffeomorphisms $\varphi = \mathrm{Id}_\mathbb{R} + f$ with $f \in \mathcal{B}(\mathbb{R})$ such that $f' > -1$. $\mathcal{B}(\mathbb{R})$ is the space of functions which have all derivatives (separately) bounded. It is a reflexive nuclear Fréchet space.
The space $C^\infty(\mathbb{R},\mathcal{B}(\mathbb{R}))$ of smooth curves $t \mapsto f(t,\cdot)$ in $\mathcal{B}(\mathbb{R})$ consists of all functions $f \in C^\infty(\mathbb{R}^2,\mathbb{R})$ satisfying the following property:

- *For all $k \in \mathbb{N}_{\geq 0}$ and $n \in \mathbb{N}_{\geq 0}$ the expression $\partial_t^k \partial_x^n f(t,x)$ is uniformly bounded in $x \in \mathbb{R}$, locally in t.*

We can specify other regular Lie groups by requiring that g lies in certain spaces of smooth functions. Now we will discuss these spaces, describe the smooth curves in them, and describe the corresponding groups, specializing the results from [38], where most of these groups are treated on $\mathbb{R}^n$ in full detail.

2.2. **Groups related to** $\mathrm{Diff}_c(\mathbb{R})$**.** The reflexive nuclear (LF) space $C^\infty_c(\mathbb{R})$ of smooth functions with compact support leads to the well-known regular Lie group $\mathrm{Diff}_c(\mathbb{R})$, see [20, 43.1].

We will now define an extension of this group which will play a major role in the later parts of this article.

Define $C^\infty_{c,2}(\mathbb{R}) = \{f : f' \in C^\infty_c(\mathbb{R})\}$ to be the space of antiderivatives of smooth functions with compact support. It is a reflexive nuclear (LF) space. We also define the space $C^\infty_{c,1}(\mathbb{R}) = \left\{f \in C^\infty_{c,2}(\mathbb{R}) \ :\ f(-\infty) = 0\right\}$ of antiderivatives of the form $x \mapsto \int_{-\infty}^x g\, dy$ with $g \in C^\infty_c(\mathbb{R})$.
$\mathrm{Diff}_{c,2}(\mathbb{R}) = \left\{\varphi = \mathrm{Id} + f : f \in C^\infty_{c,2}(\mathbb{R}), f' > -1\right\}$ *is the corresponding group.*

Define the two functionals $\mathrm{Shift}_\ell, \mathrm{Shift}_r : \mathrm{Diff}_{c,2}(\mathbb{R}) \to \mathbb{R}$ by

$$\mathrm{Shift}_\ell(\varphi) = \mathrm{ev}_{-\infty}(f) = \lim_{x\to-\infty} f(x), \quad \mathrm{Shift}_r(\varphi) = \mathrm{ev}_{\infty}(f) = \lim_{x\to\infty} f(x)$$

for $\varphi(x) = x + f(x)$. Then the short exact sequence of smooth homomorphisms of Lie groups

$$\mathrm{Diff}_c(\mathbb{R}) \rightarrowtail \mathrm{Diff}_{c,2}(\mathbb{R}) \xrightarrow{(\mathrm{Shift}_\ell, \mathrm{Shift}_r)} (\mathbb{R}^2, +)$$

describes a semidirect product, where a smooth homomorphic section $s : \mathbb{R}^2 \to \mathrm{Diff}_{c,2}(\mathbb{R})$ is given by the composition of flows $s(a,b) = \mathrm{Fl}^{X_\ell}_a \circ \mathrm{Fl}^{X_r}_b$ for the vectorfields $X_\ell = f_\ell \partial_x$, $X_r = f_r\partial_x$ with $[X_\ell, X_r] = 0$ where $f_\ell, f_r \in C^\infty(\mathbb{R}, [0,1])$ satisfy

$$f_\ell(x) = \begin{cases} 1 & \text{for } x \le -1 \\ 0 & \text{for } x \ge 0, \end{cases} \qquad f_r(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 & \text{for } x \ge 1. \end{cases} \tag{2}$$

The normal subgroup $\mathrm{Diff}_{c,1}(\mathbb{R}) = \ker(\mathrm{Shift}_\ell) = \{\varphi = \mathrm{Id} + f : f \in C^\infty_{c,1}(\mathbb{R}), f' > -1\}$ of diffeomorphisms which have no shift at $-\infty$ will play an important role later on.

2.3. **Groups related to $\mathrm{Diff}_\mathcal{S}(\mathbb{R})$.** The regular Lie group $\mathrm{Diff}_\mathcal{S}(\mathbb{R})$ was treated in [33, 6.4]. Let us summarize the most important results: The space $\mathcal{S}(\mathbb{R})$ consisting of all rapidly decreasing functions is a reflexive nuclear Fréchet space.
The space $C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}))$ of smooth curves in $\mathcal{S}(\mathbb{R})$ consists of all functions $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ satisfying the following property:

- *For all $k, m \in \mathbb{N}_{\ge 0}$ and $n \in \mathbb{N}_{\ge 0}$, the expression $(1 + |x|^2)^m \partial_t^k \partial_x^n f(t,x)$ is uniformly bounded in $x \in \mathbb{R}$, locally uniformly bounded in $t \in \mathbb{R}$.*

$\mathrm{Diff}_\mathcal{S}(\mathbb{R}) = \{\varphi = \mathrm{Id} + f : f \in \mathcal{S}(\mathbb{R}), f' > -1\}$ *is the corresponding regular Lie group.*

We again define an extended space:

$\mathcal{S}_2(R) = \{f \in C^\infty(\mathbb{R}) : f' \in \mathcal{S}(\mathbb{R})\}$, the space of antiderivatives of functions in $\mathcal{S}(\mathbb{R})$. It is isomorphic to $\mathbb{R} \times \mathcal{S}(\mathbb{R})$ via $f \mapsto (f(0), f')$. It is again a reflexive nuclear Fréchet space, contained in $\mathcal{B}(\mathbb{R})$.
The space $C^\infty(\mathbb{R}, \mathcal{S}_2(\mathbb{R}))$ of smooth curves in $\mathcal{S}_2(\mathbb{R})$ consists of all functions $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ satisfying the following property:

- *For all $k, m \in \mathbb{N}_{\ge 0}$, and $n \in \mathbb{N}_{>0}$, the expression $(1 + |x|^2)^m \partial_t^k \partial_x^n f(t,x)$ is uniformly bounded in x and locally uniformly bounded in t.*

We also define the space $\mathcal{S}_1(\mathbb{R}) = \{f \in \mathcal{S}_2(\mathbb{R}) : f(-\infty) = 0\}$ of antiderivatives of the form $x \mapsto \int_{-\infty}^x g\, dy$ with $g \in \mathcal{S}(\mathbb{R})$.

$\mathrm{Diff}_{\mathcal{S}_2}(\mathbb{R}) = \{\varphi = \mathrm{Id} + f : f \in \mathcal{S}_2(\mathbb{R}), f' > -1\}$ *is the corresponding regular Lie group.* We have again the short exact sequence of smooth homomorphisms of Lie groups

$$\mathrm{Diff}_\mathcal{S}(\mathbb{R}) \rightarrowtail \mathrm{Diff}_{\mathcal{S}_2}(\mathbb{R}) \xrightarrow{(\mathrm{Shift}_\ell, \mathrm{Shift}_r)} (\mathbb{R}^2, +)$$

which splits via the same smooth homomorphic section $s : \mathbb{R}^2 \to \mathrm{Diff}_{\mathcal{S}_2}(\mathbb{R})$ as in 2.4 and thus describes a semi-direct product. The normal Lie subgroup $\mathrm{Diff}_{\mathcal{S}_1}(\mathbb{R}) = \ker(\mathrm{Shift}_\ell)$ of diffeomorphisms which have no shift at $-\infty$ will also play an important role later on.

2.4. **Groups related to** $\mathrm{Diff}_{H^\infty}(\mathbb{R})$**.** The space $H^\infty(\mathbb{R}) = \bigcap_{k\geq 0} H^k(\mathbb{R})$ is the intersection of all Sobolev spaces. It is a reflexive Fréchet space. For any $f \in H^\infty(\mathbb{R})$ each derivative $f^{(k)}$ is smooth and converges to 0 for $x \to \pm\infty$ by the Lemma of Riemann-Lebesgue. But $H^\infty(\mathbb{R}) \not\subset L^1(\mathbb{R})$: a smooth function which equals $1/|x|$ for large $|x|$ is in $H^\infty(\mathbb{R})$ but not in $L^1(\mathbb{R})$.
The space $C^\infty(\mathbb{R}, H^\infty(\mathbb{R}))$ of smooth curves $t \mapsto f(t,\cdot)$ in $H^\infty(\mathbb{R})$ consists of all functions $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ satisfying the following property:

- *For all $k \in \mathbb{N}_{\geq 0}$ and $n \in \mathbb{N}_{\geq 0}$ the expression $\|\partial_t^k \partial_x^n f(t, \quad)\|_{L^2(\mathbb{R})}$ is locally bounded in t.*

$\mathrm{Diff}_{H^\infty}(\mathbb{R}) = \big\{\varphi = \mathrm{Id} + f : f \in H^\infty(\mathbb{R}), f' > -1\big\}$ *denotes the corresponding regular Lie group.*

We again consider an extended space:

$H_2^\infty(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : f' \in H^\infty(\mathbb{R}) \cap L^1(\mathbb{R})\}$ is the space of bounded antiderivatives of functions in $H^\infty(\mathbb{R})$. It is isomorphic to $\mathbb{R} \times H^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ via $f \mapsto (f(0), f')$. *The space $C^\infty(\mathbb{R}, H_2^\infty(\mathbb{R}))$ of smooth curves in $H^\infty(\mathbb{R})$ consists of all functions $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ satisfying the following property:*

- *For all $k \in \mathbb{N}_{\geq 0}$, $n \in \mathbb{N}_{>0}$ and $t \in \mathbb{R}$ the expressions $\|\partial_t^k \partial_x^n f(t, \quad)\|_{L^2(\mathbb{R})}$ and $\|\partial_t^k f(t, \quad)\|_{L^1}$ are locally bounded in t.*

$\mathrm{Diff}_{H_2^\infty}(\mathbb{R}) = \big\{\varphi = \mathrm{Id} + f : f \in H^\infty(\mathbb{R}), f' > -1\big\}$ *denotes the corresponding regular Lie group.*

In contrast to the previous cases – $\mathrm{Diff}_{c,2}(\mathbb{R})$ and $\mathrm{Diff}_{\mathcal{S}_2}(\mathbb{R})$ – the space $\mathrm{Diff}_{H_2^\infty}(\mathbb{R})$ is a semidirect product of $\mathbb{R}^2$ with a normal subgroup $\mathrm{Diff}_{H_0^\infty}(\mathbb{R})$ which is larger than $\mathrm{Diff}_{H^\infty}(\mathbb{R})$. To see this we fist consider the exact sequence of Fréchet spaces

$$H_0^\infty(\mathbb{R}) \rightarrowtail H_2^\infty(\mathbb{R}) \xrightarrow{(\mathrm{ev}_{-\infty}, \mathrm{ev}_\infty)} \mathbb{R}^2 \quad .$$

where $H_0^\infty(\mathbb{R})$ is just the kernel of $(\mathrm{ev}_{-\infty}, \mathrm{ev}_\infty)$. The space $H_0^\infty(\mathbb{R})$ contains $H^\infty(\mathbb{R})$ but is larger. In fact,

$$H_0^\infty(\mathbb{R}) = \Big\{x \mapsto \int_{-\infty}^x g(y)dy : g \in H^\infty(\mathbb{R}) \cap L^1(\mathbb{R}), \int_{-\infty}^\infty g(x)dx = 0\Big\}.$$

Let $f \in C^\infty(\mathbb{R}, \mathbb{R})$ be a smooth odd function such that $f(x) = x^{-\frac{3}{2}}$ for $x < -1$. Then $f \in H^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\int_{-\infty}^\infty f(x)dx = 0$ because f is odd, but for $x < -1$ the function $x \mapsto \int_{-\infty}^x f(y)dy = -\frac{1}{2}x^{-\frac{1}{2}}$ is not square-integrable and thus $x \mapsto \int_{-\infty}^x f(y)dy \notin H^\infty(\mathbb{R})$. But still, $H_0^\infty(\mathbb{R}) \subset \mathcal{B}(\mathbb{R})$ is an ideal; this is not proved directly in [38], but the proofs there can easily be adapted. We define the group

$$\mathrm{Diff}_{H_0^\infty}(\mathbb{R}) := \big\{\mathrm{Id} + f : f \in H_0^\infty(\mathbb{R}), f' > -1\big\}.$$

In the following sequence

$$\mathrm{Diff}_{H^\infty}(\mathbb{R}) \subsetneq \mathrm{Diff}_{H_0^\infty}(\mathbb{R}) \rightarrowtail \mathrm{Diff}_{H_2^\infty}(\mathbb{R}) \xrightarrow{(\mathrm{Shift}_\ell, \mathrm{Shift}_r)} (\mathbb{R}^2, +) \tag{3}$$

only the right hand side triple is exact and splitting with the same sections as for $\mathrm{Diff}_{c,2}(\mathbb{R})$.

The space $\mathrm{Diff}_{H_1^\infty}(\mathbb{R}) = \ker(\mathrm{Shift}_\ell)$ of diffeomorphisms, which have no shift at $-\infty$, will also play an important role later on.

2.5. **Theorem.** *The groups* $\mathrm{Diff}_c(\mathbb{R})$, $\mathrm{Diff}_{c,1}(\mathbb{R})$, $\mathrm{Diff}_{c,2}(\mathbb{R})$, $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$, $\mathrm{Diff}_{\mathcal{S}_1}(\mathbb{R})$, $\mathrm{Diff}_{\mathcal{S}_2}(\mathbb{R})$, $\mathrm{Diff}_{H^\infty}(\mathbb{R})$, $\mathrm{Diff}_{H_0^\infty}(\mathbb{R})$, $\mathrm{Diff}_{H_1^\infty}(\mathbb{R})$, $\mathrm{Diff}_{H_2^\infty}(\mathbb{R})$, *and* $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R})$ *are all smooth regular Lie groups. We have the following smooth injective group homomorphisms:*

$$\begin{tikzcd}
 & & \mathrm{Diff}_{H^\infty}(\mathbb{R}) \arrow[d] & \\
\mathrm{Diff}_c(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}) \arrow[r] \arrow[d] \arrow[ur] & \mathrm{Diff}_{H_0^\infty}(\mathbb{R}) \arrow[d] & \\
\mathrm{Diff}_{c,1}(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{Diff}_{\mathcal{S}_1}(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{Diff}_{H_1^\infty}(\mathbb{R}) \arrow[d] & \\
\mathrm{Diff}_{c,2}(\mathbb{R}) \arrow[r] & \mathrm{Diff}_{\mathcal{S}_2}(\mathbb{R}) \arrow[r] & \mathrm{Diff}_{H_2^\infty}(\mathbb{R}) \arrow[r] & \mathrm{Diff}_{\mathcal{B}}(\mathbb{R})
\end{tikzcd}$$

Each group is a normal subgroup in any other in which it is contained, in particular in $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R})$.

Proof. That the the groups $\mathrm{Diff}_c(\mathbb{R})$, $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$, $\mathrm{Diff}_{H^\infty}(\mathbb{R})$, and $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R})$ are regular Lie groups is proved (for $\mathbb{R}^n$ instead of $\mathbb{R}$) in [38] and in [20, 43.1] for $\mathrm{Diff}_c(\mathbb{R})$. The proofs there can be adapted to the case $\mathrm{Diff}_{H_0^\infty}(\mathbb{R})$. The extension to the semi-direct products is easy and is proved in [20, 38.9]. That each group is normal in the largest one is also proved in [38]. □

2.6. **A remark on the existence of normal subgroups.** This section will not be used in the rest of the paper. It is a well known result that $\mathrm{Diff}_c(\mathbb{R})$ is a simple group; see [29], [30], [31]. We want to discuss some effects of this result for the diffeomorphism groups introduced in the previous sections.

Existence of normal subgroups in $\mathrm{Diff}_{c,2}(\mathbb{R})$*:* We first claim that any non-trivial normal subgroup N of $\mathrm{Diff}_{c,2}(\mathbb{R})$ intersects $\mathrm{Diff}_c(\mathbb{R})$ nontrivially: Let $\mathrm{Id} \neq \varphi \in N$. If φ has compact support we are done. If φ does not have compact support, without loss of generality assume that $\mathrm{Shift}_r(\varphi) = a > 0$. Thus for some x_0 we have $\varphi(x) = x + a$ for $x \geq x_0$. Choose $x_0 + 2a < x_1 < x_2 < x_0 + 3a$ and $\psi \in \mathrm{Diff}_c(\mathbb{R})$ with support in the interval $[x_1, x_2]$, so that $\psi(x) = x$ for $x \notin [x_1, x_2]$. For $x \in [x_1, x_2]$ we then have $(\psi^{-1} \circ \varphi \circ \psi)(x) = \psi^{-1}(\psi(x) + a) = \psi(x) + a$. Thus $\psi^{-1} \circ \varphi \circ \psi \in N$ differs from φ just on the compact intervall $[x_1, x_2]$. But then $\psi^{-1} \circ \varphi \circ \psi \circ \varphi^{-1} \in N$ has compact support and we are done.

By simplicity of $\mathrm{Diff}_c(\mathbb{R})$ we get $N \supseteq \mathrm{Diff}_c(\mathbb{R})$. Therefore the lattice of normal subgroups of $\mathrm{Diff}_{c,2}(\mathbb{R})$ has a $\mathrm{Diff}_c(\mathbb{R})$ as minimal element, and thus is isomorphic to the lattice of subgroups of $(\mathbb{R}^2, +)$, which is quite large; see [11], [12].

Existence of normal subgroups N *with* $\mathrm{Diff}_c(\mathbb{R}) \to N \to \mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$*:* By conjugating with $x \mapsto 1/x$ we see that the quotient group $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})/\mathrm{Diff}_c(\mathbb{R})$ is isomorphic to the group of germs at 0 of smooth diffeomorphisms $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\varphi(x) - x$ is flat at 0: $\varphi(x) - x = o(|x|^N)$ for each N. This group contains infinitely many normal subgroups: $\varphi(x) - x = o(e^{-1/|x|})$ or $= o(\exp(-\exp(1/|x|)))$, and so on.

We have not looked for normal subgroups N with $\mathrm{Diff}_{H^\infty}(\mathbb{R}) \to N \to \mathrm{Diff}_{H_0^\infty}(\mathbb{R})$.

2.7. **Groups of real analytic diffeomorphisms.** Since the Hunter-Saxton equation will turn out to have solutions in smaller groups of diffeomorphisms, we give here a description of them. For simplicity's sake, we restrict our attention to the periodic case. Let $\mathrm{Diff}^\omega(S^1)$ be the real analytic regular Lie group of all real analytic diffeomorphisms of S^1, with the real analytic structure described in [18, 8.11], see also [20, theorem 43.4]. Let us just recall here that a mapping between c^∞-open sets of convenient vector spaces is real analytic if preserves smooth curves and also real analytic curves.

2.8. **Groups of ultra-differentiable diffeomorphisms.** Let us now describe the Denjoy-Carleman ultra-differentiable function classes which admit convenient calculus, following [22], [23], and [24]. We consider a sequence $M = (M_k)$ of positive real numbers serving as weights for derivatives. For a smooth function f on an open subset U in $\mathbb{R}^n$, a compact set $K \subset U$, and for $\rho > 0$ consider the set

$$\left\{ \frac{d^k f(x)}{\rho^k \, k! \, M_k} : x \in K, k \in \mathbb{N} \right\} . \tag{4}$$

We define the *Denjoy–Carleman classes*

$$C^{(M)}(U) := \{ f \in C^\infty(U) : \forall \text{ compact } K \subseteq U \ \forall \rho > 0 : (4) \text{ is bounded} \} ,$$
$$C^{\{M\}}(U) := \{ f \in C^\infty(U) : \forall \text{ compact } K \subseteq U \ \exists \rho > 0 : (4) \text{ is bounded} \} .$$

The elements of $C^{(M)}(U)$ are said to be of *Beurling type*; those of $C^{\{M\}}(U)$ of *Roumieu type*. If $M_k = 1$, for all k, then $C^{(M)}(\mathbb{R})$ consists of the restrictions to U of the real and imaginary parts of all entire functions, while $C^{\{M\}}(\mathbb{R})$ coincides with the ring $C^\omega(\mathbb{R})$ of real analytic functions. We shall also write $C^{[M]}$ if mean either $C^{(M)}$ or $C^{\{M\}}$. We shall assume that the sequence $M = (M_k)$ has the following properties:

- M is log-convex: $k \mapsto \log(M_k)$ is convex, i.e., $M_k^2 \le M_{k-1} \, M_{k+1}$ for all k .
- M has moderate growth, i.e., $\sup_{j,k \in \mathbb{N}_{>0}} \left(\frac{M_{j+k}}{M_j \, M_k} \right)^{\frac{1}{j+k}} < \infty$.
- In the Beurling case $C^{[M]} = C^{(M)}$ we also require that $C^\omega \subseteq C^{(M)}$, or equivalently $M_k^{1/k} \to \infty$.

Then, both classes $C^{[M]}$ (2) are closed under composition and differentiation, (3) can be extended to to convenient vector spaces, and (4) form monoidally closed categories (i.e., admit convenient settings). Moreover, (5) on open sets in $\mathbb{R}^n$, $C^{[M]}$-vector fields have $C^{[M]}$-flows, and (6) between Banach spaces, the $C^{[M]}$ implicit function theorem holds.

For mappings between c^∞-open subsets of convenient vector spaces we have:

- For non-quasianalytic M, the mapping f is $C^{\{M\}}$ if it maps $C^{\{M\}}$-curves to $C^{\{M\}}$-curves, by [22].
- For certain quasianalytic M, the mapping f is $C^{\{M\}}$ if it maps $C^{\{N\}}$-curves to $C^{\{N\}}$-curves, for all non-quasianalytic N which are larger than M and have the above properties, by [23].
- For any M, the mapping f is $C^{[M]}$ if it respects $C^{[M]}$-maps from open balls in Banach spaces, by [24].

For every M with properties above we have the regular Lie groups $\mathrm{Diff}^{\{M\}}(S^1)$ and $\mathrm{Diff}^{[M]}(S^1)$ (we write $\mathrm{Diff}^{[M]}(S^1)$ if we mean any of the two) of $C^{[M]}$-diffeomorphisms of S^1 which is a $C^{[M]}$-group (but not better), by [22, 6.5], [23, 5.6], and [24, 9.8].

3. Right invariant Riemannian metrics on Lie groups

3.1. **Notation on Lie groups.** Let G be a regular Lie group, which may be infinite dimensional, with Lie algebra $\mathfrak{g}$. Let $\mu : G \times G \to G$ be the group multiplication, μ_x the left translation and μ^y the right translation, given by $\mu_x(y) = \mu^y(x) = xy = \mu(x, y)$.

Let $L, R : \mathfrak{g} \to \mathfrak{X}(G)$ be the left and right invariant vector field mappings, given by $L_X(g) = T_e(\mu_g).X$ and $R_X = T_e(\mu^g).X$ respectively. They are related by $L_X(g) = R_{\mathrm{Ad}(g)X}(g)$. Their flows are given by

$$\mathrm{Fl}_t^{L_X}(g) = g.\exp(tX) = \mu^{\exp(tX)}(g), \quad \mathrm{Fl}_t^{R_X}(g) = \exp(tX).g = \mu_{\exp(tX)}(g)\ .$$

We also need the right Maurer-Cartan form $\kappa = \kappa^r \in \Omega^1(G, \mathfrak{g})$, given by $\kappa_x(\xi) := T_x(\mu^{x^{-1}}) \cdot \xi$. It satisfies the left Maurer-Cartan equation $d\kappa - \frac{1}{2}[\kappa, \kappa]_\wedge = 0$, where $[\ \ ,\ \ ]_\wedge$ denotes the wedge product of $\mathfrak{g}$-valued forms on G induced by the Lie bracket. Note that $\frac{1}{2}[\kappa, \kappa]_\wedge(\xi, \eta) = [\kappa(\xi), \kappa(\eta)]$. The (exterior) derivative of the function $\mathrm{Ad} : G \to GL(\mathfrak{g})$ can be expressed by

$$d\,\mathrm{Ad} = \mathrm{Ad}\,.(\mathrm{ad} \circ \kappa^l) = (\mathrm{ad} \circ \kappa^r).\,\mathrm{Ad}\ ,$$

since we have $d\,\mathrm{Ad}(T\mu_g.X) = \frac{d}{dt}|_0 \mathrm{Ad}(g.\exp(tX)) = \mathrm{Ad}(g).\,\mathrm{ad}(\kappa^l(T\mu_g.X))$.

3.2. **Geodesics of a right invariant metric on a Lie group.** Let $\gamma = \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a positive definite bounded (weak) inner product. Then

$$\gamma_x(\xi, \eta) = \gamma\big(T(\mu^{x^{-1}}) \cdot \xi,\, T(\mu^{x^{-1}}) \cdot \eta\big) = \gamma\big(\kappa(\xi),\, \kappa(\eta)\big) \tag{5}$$

is a right invariant (weak) Riemannian metric on G, and any (weak) right invariant bounded Riemannian metric is of this form, for suitable γ. We shall denote by $\check{\gamma} : \mathfrak{g} \to \mathfrak{g}^*$ the mapping induced by γ from the Lie algebra into its dual (of bounded linear functionals) and by $\langle \alpha, X \rangle_{\mathfrak{g}}$ the duality evaluation between $\alpha \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.

Let $g : [a, b] \to G$ be a smooth curve. The velocity field of g, viewed in the right trivializations, coincides with the right logarithmic derivative

$$\delta^r(g) = T(\mu^{g^{-1}}) \cdot \partial_t g = \kappa(\partial_t g) = (g^*\kappa)(\partial_t), \text{ where } \partial_t = \frac{\partial}{\partial t}\ .$$

The energy of the curve $g(t)$ is given by

$$E(g) = \tfrac{1}{2} \int_a^b \gamma_g(g', g')dt = \tfrac{1}{2} \int_a^b \gamma\big((g^*\kappa)(\partial_t), (g^*\kappa)(\partial_t)\big)\, dt\ . \tag{6}$$

For a variation $g(s, t)$ with fixed endpoints we use then that

$$d(g^*\kappa)(\partial_t, \partial_s) = \partial_t(g^*\kappa(\partial_s)) - \partial_s(g^*\kappa(\partial_t)) - 0\ ,$$

partial integration, and the left Maurer-Cartan equation to obtain:

$$\begin{aligned}
\partial_s E(g) &= \tfrac{1}{2}\int_a^b 2\gamma\big(\partial_s(g^*\kappa)(\partial_t),\,(g^*\kappa)(\partial_t)\big)\,dt\\
&= \int_a^b \gamma\big(\partial_t(g^*\kappa)(\partial_s) - d(g^*\kappa)(\partial_t,\partial_s),\,(g^*\kappa)(\partial_t)\big)\,dt\\
&= -\int_a^b \gamma\big((g^*\kappa)(\partial_s),\,\partial_t(g^*\kappa)(\partial_t)\big)\,dt - \int_a^b \gamma\big([(g^*\kappa)(\partial_t),(g^*\kappa)(\partial_s)],\,(g^*\kappa)(\partial_t)\big)\,dt\\
&= -\int_a^b \big\langle\check\gamma(\partial_t(g^*\kappa)(\partial_t)),\,(g^*\kappa)(\partial_s)\big\rangle_{\mathfrak g}\,dt\\
&\qquad - \int_a^b \big\langle\check\gamma((g^*\kappa)(\partial_t)),\,\mathrm{ad}_{(g^*\kappa)(\partial_t)}(g^*\kappa)(\partial_s)\big\rangle_{\mathfrak g}\,dt\\
&= -\int_a^b \big\langle\check\gamma(\partial_t(g^*\kappa)(\partial_t)) + (\mathrm{ad}_{(g^*\kappa)(\partial_t)})^*\check\gamma((g^*\kappa)(\partial_t)),\,(g^*\kappa)(\partial_s)\big\rangle_{\mathfrak g}\,dt\,.
\end{aligned}$$

Thus the curve $g(0,t)$ is critical for the energy (6) if and only if

$$\check\gamma(\partial_t(g^*\kappa)(\partial_t)) + (\mathrm{ad}_{(g^*\kappa)(\partial_t)})^*\check\gamma((g^*\kappa)(\partial_t)) = 0\,.$$

In terms of the right logarithmic derivative $u:[a,b]\to\mathfrak g$ of $g:[a,b]\to G$, given by $u(t) := g^*\kappa(\partial_t) = T_{g(t)}(\mu^{g(t)^{-1}})\cdot g'(t)$, the geodesic equation has the expression:

$$\boxed{\partial_t u = -\,\check\gamma^{-1}\,\mathrm{ad}(u)^*\,\check\gamma(u)} \tag{7}$$

Thus the geodesic equation exists in general if and only if $\mathrm{ad}(X)^*\check\gamma(X)$ is in the image of $\check\gamma:\mathfrak g\to\mathfrak g^*$, i.e.

$$\mathrm{ad}(X)^*\check\gamma(X)\in\check\gamma(\mathfrak g) \tag{8}$$

for every $X\in\mathfrak X$. The condition (8) then leads to the existence of the 'Christoffel symbols'. Interestingly it is not neccessary for the more restrictive condition $\mathrm{ad}(X)^*\check\gamma(Y)\in\check\gamma\in\mathfrak g$ to be satisfied in order to obtain the geodesic equation, Christoffel symbols and the curvature; compare with [33, Lemma 3.3]. Note here the appearance of the geodesic equation for the *momentum* $p:=\gamma(u)$:

$$p_t = -\,\mathrm{ad}(\check\gamma^{-1}(p))^*p\,.$$

We shall meet situations later on where (8) is satisfied but where the usual transpose $\mathrm{ad}^\top(X)$ of $\mathrm{ad}(X)$,

$$\mathrm{ad}^\top(X) := \check\gamma^{-1}\circ\mathrm{ad}^*_X\circ\check\gamma \tag{9}$$

does not exist for all X.

3.3. **The covariant derivative.** The right trivialization $(\pi_G,\kappa^r): TG\to G\times\mathfrak g$ induces the isomorphism $R: C^\infty(G,\mathfrak g)\to\mathfrak X(G)$, given by $R(X)(x) := R_X(x) := T_e(\mu^x)\cdot X(x)$, for $X\in C^\infty(G,\mathfrak g)$ and $x\in G$. Here $\mathfrak X(G):=\Gamma(TG)$ denotes the Lie algebra of all vector fields. For the Lie bracket and the Riemannian metric we have

$$\begin{aligned}
[R_X,R_Y] &= R(-[X,Y]_{\mathfrak g} + dY\cdot R_X - dX\cdot R_Y)\,,\\
R^{-1}[R_X,R_Y] &= -[X,Y]_{\mathfrak g} + R_X(Y) - R_Y(X)\,,\\
\gamma_x(R_X(x),R_Y(x)) &= \gamma(X(x),Y(x))\,,\ x\in G\,.
\end{aligned}$$

In the following we shall perform all computations in $C^\infty(G,\mathfrak g)$ instead of $\mathfrak X(G)$. In particular, we shall use the convention

$$\nabla_X Y := R^{-1}(\nabla_{R_X}R_Y)\quad\text{ for } X,Y\in C^\infty(G,\mathfrak g)\,.$$

to express the Levi-Civita covariant derivative.

Lemma. *Assume that for all $\xi \in \mathfrak{g}$ the element $\operatorname{ad}(\xi)^*\check\gamma(\xi) \in \mathfrak{g}^*$ is in the image of $\check\gamma : \mathfrak{g} \to \mathfrak{g}^*$ and that $\xi \mapsto \check\gamma^{-1}\operatorname{ad}(\xi)^*\check\gamma(\xi)$ is bounded quadratic (equivalently, smooth). Then the Levi-Civita covariant derivative of the metric γ exists and is given for any $X, Y \in C^\infty(G, \mathfrak{g})$ in terms of the isomorphism R by*

$$\nabla_X Y = dY.R_X + \rho(X)Y - \tfrac{1}{2}\operatorname{ad}(X)Y \ ,$$

where

$$\rho(\xi)\eta = \tfrac{1}{4}\check\gamma^{-1}\big(\operatorname{ad}^*_{\xi+\eta}\check\gamma(\xi+\eta) - \operatorname{ad}^*_{\xi-\eta}\check\gamma(\xi-\eta)\big) = \tfrac{1}{2}\check\gamma^{-1}\big(\operatorname{ad}^*_\xi\check\gamma(\eta) + \operatorname{ad}^*_\eta\check\gamma(\xi)\big)$$

is the polarized version. $\rho : \mathfrak{g} \to L(\mathfrak{g}, \mathfrak{g})$ is bounded and we have $\rho(\xi)\eta = \rho(\eta)\xi$. We also have:

$$\gamma\big(\rho(\xi)\eta, \zeta\big) = \tfrac{1}{2}\gamma(\xi, \operatorname{ad}(\eta)\zeta) + \tfrac{1}{2}\gamma(\eta, \operatorname{ad}(\xi)\zeta) \ ,$$
$$\gamma(\rho(\xi)\eta, \zeta) + \gamma(\rho(\eta)\zeta, \xi) + \gamma(\rho(\zeta)\xi, \xi) = 0 \ .$$

Proof. It is easily checked that ∇ is a covariant derivative. The Riemannian metric is covariantly constant, since

$$R_X\gamma(Y, Z) = \gamma(dY.R_X, Z) + \gamma(Y, dZ.R_X) = \gamma(\nabla_X Y, Z) + \gamma(Y, \nabla_X Z) \ .$$

Since ρ is symmetric, the connection is also torsionfree:

$$\nabla_X Y - \nabla_Y X + [X, Y]_{\mathfrak{g}} - dY.R_X + dX.R_Y = 0.$$

□

3.4. **The curvature.** For $X, Y \in C^\infty(G, \mathfrak{g})$ we have

$$[R_X, \operatorname{ad}(Y)] = \operatorname{ad}(R_X(Y)) \quad \text{and} \quad [R_X, \rho(Y)] = \rho(R_X(Y)).$$

The Riemannian curvature is then computed by

$$\begin{aligned}
\mathcal{R}(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{-[X,Y]_{\mathfrak{g}} + R_X(Y) - R_Y(X)} \\
&= [R_X + \rho_X - \tfrac{1}{2}\operatorname{ad}_X, R_Y + \rho_Y - \tfrac{1}{2}\operatorname{ad}_Y] \\
&\quad - R(-[X, Y]_{\mathfrak{g}} + R_X(Y) - R_Y(X)) \\
&\quad - \rho(-[X, Y]_{\mathfrak{g}} + R_X(Y) - R_Y(X)) \\
&\quad + \tfrac{1}{2}\operatorname{ad}(-[X, Y]_{\mathfrak{g}} + R_X(Y) - R_Y(X)) \\
&= [\rho_X, \rho_Y] + \rho_{[X,Y]_{\mathfrak{g}}} - \tfrac{1}{2}[\rho_X, \operatorname{ad}_Y] + \tfrac{1}{2}[\rho_Y, \operatorname{ad}_X] - \tfrac{1}{4}\operatorname{ad}_{[X,Y]_{\mathfrak{g}}} \ .
\end{aligned}$$

which visibly is a tensor field.

3.5. **The sectional curvature.** For the linear 2-dimensional subspace $P \subseteq \mathfrak{g}$ spanned by linearly independent $X, Y \in \mathfrak{g}$, the sectional curvature is defined as:

$$k(P) = -\frac{\gamma\big(\mathcal{R}(X, Y)X, Y\big)}{\|X\|^2_\gamma \|Y\|^2_\gamma - \gamma(X, Y)^2} \ .$$

For the numerator we get:

$$\begin{aligned}
\gamma\big(\mathcal{R}(X, Y)X, Y\big) &= \gamma(\rho_X\rho_Y X, Y) - \gamma(\rho_Y\rho_X X, Y) + \gamma(\rho_{[X,Y]}X, Y) \\
&\quad - \tfrac{1}{2}\gamma(\rho_X[Y, X], Y) + \tfrac{1}{2}\gamma([Y, \rho_X X], Y) \\
&\quad + 0 - \tfrac{1}{2}\gamma([X, \rho_Y X], Y) - \tfrac{1}{4}\gamma([[X, Y], X], Y) \\
&= \gamma(\rho_X X, \rho_Y Y) - \|\rho_X Y\|^2_\gamma + \tfrac{3}{4}\|[X, Y]\|^2_\gamma \\
&\quad - \tfrac{1}{2}\gamma(X, [Y, [X, Y]]) + \tfrac{1}{2}\gamma(Y, [X, [X, Y]]). \\
&= \gamma(\rho_X X, \rho_Y Y) - \|\rho_X Y\|^2_\gamma + \tfrac{3}{4}\|[X, Y]\|^2_\gamma \\
&\quad - \gamma(\rho_X Y, [X, Y]]) + \gamma(Y, [X, [X, Y]]) \ .
\end{aligned}$$

If the adjoint $\mathrm{ad}(X)^\top : \mathfrak{g} \to \mathfrak{g}$ exists, this is easily seen to coincide with Arnold's original formula [1],

$$\gamma(\mathcal{R}(X,Y)X,Y) = -\tfrac{1}{4}\| \mathrm{ad}(X)^\top Y + \mathrm{ad}(Y)^\top X\|_\gamma^2 + \gamma(\mathrm{ad}(X)^\top X, \mathrm{ad}(Y)^\top Y)$$
$$+ \tfrac{1}{2}\gamma(\mathrm{ad}(X)^\top Y - \mathrm{ad}(Y)^\top X, \mathrm{ad}(X)Y) + \tfrac{3}{4}\|[X,Y]\|_\gamma^2 \ .$$

4. The homogeneous H^1-metric on Diff($\mathbb{R}$) and the Hunter-Saxton equation

In this section we will study the homogeneous H^1 or $\dot{H}^1$-metric on the various diffeomorphism groups of $\mathbb{R}$ defined in Section 2. It was shown in [16] that the geodesic equation of the $\dot{H}^1$-metric on $\mathrm{Diff}(S^1)$ is the Hunter-Saxton equation. We will show that suitable diffeomorphism groups on the real line also have the HS-equation as geodesic equation. In [26] a way was found to map $\mathrm{Diff}(S^1)$ isometrically to an open subset of an L^2-sphere in $C^\infty(S^1,\mathbb{R})$. This representation we will generalize to the non-periodic case.

In the situation studied here – diffeomorphism groups on the real line – the resulting geometry will be different from the periodic case. Some of the diffeomorphism groups will be flat in the sense of Riemannian geometry, while others will be submanifolds of a flat space, see [18, Sect. 27.11] for the definiton of (splitting) submanifolds in an infinite dimensional setting.

4.1. **The setting.** In this section, we shall use any of the following regular Lie groups:

(1) We will denote by $\mathcal{A}(\mathbb{R})$ any of the spaces $C_c^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$ or $H^\infty(\mathbb{R})$ and by $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ the corresponding groups $\mathrm{Diff}_c(\mathbb{R})$, $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$ or $\mathrm{Diff}_{H^\infty}(\mathbb{R})$ as defined in Sections 2.2, 2.3 and 2.4.
(2) Similarly $\mathcal{A}_1(\mathbb{R})$ will denote any of the spaces $C_{c,1}^\infty(\mathbb{R})$, $\mathcal{S}_1(\mathbb{R})$ or $H_1^\infty(\mathbb{R})$ and $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ the corresponding groups $\mathrm{Diff}_{c,1}(\mathbb{R})$, $\mathrm{Diff}_{\mathcal{S}_1}(\mathbb{R})$ or $\mathrm{Diff}_{H_1^\infty}(\mathbb{R})$ as defined in Sections 2.2, 2.3 and 2.4.

4.2. **The $\dot{H}^1$-metric.** For $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ and $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ the homogeneous H^1-metric is given by

$$G_\varphi(X\circ\varphi, Y\circ\varphi) = G_{\mathrm{Id}}(X,Y) = \int_{\mathbb{R}} X'(x)Y'(x)\,dx \ ,$$

where X,Y are elements of the Lie algebra $\mathcal{A}(\mathbb{R})$ or $\mathcal{A}_1(\mathbb{R})$. We shall also use the notation

$$\langle\cdot,\cdot\rangle_{\dot{H}^1} := G(\cdot,\cdot) \ .$$

Theorem. *On $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ the geodesic equation is the Hunter-Saxton equation*

$$\boxed{(\varphi_t)\circ\varphi^{-1} = u \qquad u_t = -uu_x + \frac{1}{2}\int_{-\infty}^x (u_x(z))^2\,dz \ ,} \tag{10}$$

and the induced geodesic distance is positive.

On the other hand the geodesic equation does not exist on the subgroups $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$, since the adjoint $\mathrm{ad}(X)^*\check{G}_{\mathrm{Id}}(X)$ does not lie in $\check{G}_{\mathrm{Id}}(\mathcal{A}(\mathbb{R}))$ for all $X\in\mathcal{A}(\mathbb{R})$.

Note that this is a natural example of a non-robust Riemannian manifold in the sense of [32, 2.4].

Proof. Note that $\check{G}_{\mathrm{Id}} : \mathcal{A}_1(\mathbb{R}) \to \mathcal{A}_1(\mathbb{R})^*$ is given by $\check{G}_{\mathrm{Id}}(X) = -X''$ if we use the L^2-pairing $X \mapsto (Y \mapsto \int XY dx)$ to embed functions into the space of distributions. We now compute the adjoint of the operator $\mathrm{ad}(X)$ as defined in Section 3.2.

$$\begin{aligned}\big\langle \mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(Y), Z \big\rangle &= \check{G}_{\mathrm{Id}}(Y, \mathrm{ad}(X)Z) = G_{\mathrm{Id}}(Y, -[X,Z]) \\ &= \int_{\mathbb{R}} Y'(x)\big(X'(x)Z(x) - X(x)Z'(x)\big)'\, dx \\ &= \int_{\mathbb{R}} Z(x)\big(X''(x)Y'(x) - (X(x)Y'(x))''\big)\, dx\ .\end{aligned}$$

Therefore the adjoint as an element of $\mathcal{A}_1^*$ is given by

$$\mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(Y) = X''Y' - (XY')''\ .$$

For $X = Y$ we can rewrite this as

$$\begin{aligned}\mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(X) &= \tfrac{1}{2}\big((X'^2)' - (X^2)'''\big) = \frac{1}{2}\Big(\int_{-\infty}^{x} X'(y)^2\, dy - (X^2)'\Big)'' \\ &= \frac{1}{2}\check{G}_{\mathrm{Id}}\Big(- \int_{-\infty}^{x} X'(y)^2\, dy + (X^2)'\Big)\ .\end{aligned}$$

If $X \in \mathcal{A}_1(\mathbb{R})$ then the function $-\frac{1}{2}\int_{-\infty}^{x} X'(y)^2\, dy + \frac{1}{2}(X^2)'$ is again an element of $\mathcal{A}_1(\mathbb{R})$. This follows immediately from the definition of $\mathcal{A}_1(\mathbb{R})$. Therefore the geodesic equation exists on $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ and is given by (10).

However if $X \in \mathcal{A}(\mathbb{R})$, a neccessary condition for $\int_{-\infty}^{x}(X'(y))^2 dy \in \mathcal{A}(\mathbb{R})$ would be $\int_{-\infty}^{\infty} X'(y)^2 dy = 0$, which would imply $X' = 0$. Thus the geodesic equation does not exist on $\mathcal{A}(\mathbb{R})$.

The positivity of geodesic distance will follow from the explicit formula given in Corollary 4.4. □

Remark. One obtains the classical form of the Hunter-Saxton equation

$$u_{tx} = -uu_{xx} - \frac{1}{2}u_x^2\ ,$$

by differentiating the above geodesic equation (10). In Section 4.3 we will use a geometric argument to give an explicit solution formula, which will also imply the well-posedness of the equation. For $\mathcal{A}_1(\mathbb{R}) = H_1^\infty(\mathbb{R})$ an analytic proof of well-posedness could also be carried out similarly as in [5, Sect. 10] by adapting the arguments to $\mathbb{R}$. Furthermore we will, using this geometric trick, conclude that the curvature of the $\dot{H}^1$ metric vanishes. One can also show this statement directly, using the adaption of Arnolds formula presented in Section 3.5. From the above proof, one can easily deduce the formula for the mapping ρ:

$$\begin{aligned}\rho(X)Y &= \frac{1}{2}\check{G}^{-1}\left(\mathrm{ad}(X)^*G(Y) + \mathrm{ad}(X)^*G(Y)\right) \\ &= \frac{1}{2}\check{G}^{-1}\left(X''Y' - (XY')'' + Y''X' - (YX')''\right) \\ &= \frac{1}{2}\check{G}^{-1}\left((X'Y')' + (XY)'''\right) \\ &= \frac{1}{2}\left(- \int_{-\infty}^{x} (X'Y')\, dx + (XY)'\right)\end{aligned}$$

Using this, the desired formula for the curvature is a straightforward calculation.

Remark. For general $X \neq Y \in \mathcal{A}_1(\mathbb{R})$ we will have $\mathrm{ad}(X)^*\check{G}_{\mathrm{Id}}(X) \notin \check{G}_{\mathrm{Id}}(\mathcal{A}_1(\mathbb{R}))$. If there was a function $Z \in \mathcal{A}_1(\mathbb{R})$ such that $-Z'' = X''Y' - (XY')''$ then a necessary condition would be $0 = Z'(-\infty) = -\int_{-\infty}^{\infty} X''Y'dx$, which will in general not be satisfied. Thus the transpose of $\mathrm{ad}(X)$ as defined in (9) does not exist; only the symmetric version $X \mapsto \check{G}_{\mathrm{Id}}^{-1}(\mathrm{ad}(X)^*\check{G}_{\mathrm{Id}}(X))$ exists.

4.3. **The square root representation on** $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$**.** We will define a map R from $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ to the space

$$\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2}) = \{f \in \mathcal{A}(\mathbb{R}) \ : \ f(x) > -2\} \ ,$$

such that the pull-back of the L^2-metric on $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ is the $\dot{H}^1$-metric on the space $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$. Since $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ is an open subset of $C^\infty(\mathbb{R})$ this implies that $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ with the $\dot{H}^1$-metric is a flat space in the sense of Riemannian geometry. This is an adaptation of the square-root representation of $\mathrm{Diff}(S^1)/S^1$ used in [26]; see also Section 6, where we review this construction.

Theorem. *We define the R-map by:*

$$R : \begin{cases} \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}) \to \mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \subset \mathcal{A}(\mathbb{R}, \mathbb{R}) \\ \qquad\qquad \varphi \mapsto 2\left((\varphi')^{1/2} - 1\right) . \end{cases}$$

The R-map is invertible with inverse

$$R^{-1} : \begin{cases} \mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \to \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}) \\ \qquad\qquad \gamma \mapsto x + \dfrac{1}{4}\displaystyle\int_{-\infty}^{x} \gamma^2 + 4\gamma \, dx . \end{cases}$$

The pull-back of the flat L^2-metric via R is the $\dot{H}^1$-metric on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$*, i.e.,*

$$R^*\langle \cdot, \cdot \rangle_{L^2} = \langle \cdot, \cdot \rangle_{\dot{H}^1} .$$

Thus the space $\big(\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1\big)$ *is a flat space in the sense of Riemannian geometry.*

Here $\langle \cdot, \cdot \rangle_{L^2}$ denotes the L^2-inner product on $\mathcal{A}(\mathbb{R})$ interpreted as a Riemannian metric on $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$, that does not depend on the basepoint, i.e.

$$G_\gamma^{L^2}(h, k) = \langle h, k \rangle_{L^2} = \int_{\mathbb{R}} h(x)k(x) \, dx ,$$

for $h, k \in \mathcal{A}(\mathbb{R}) \cong T_\gamma\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$.

Proof. We will first prove that for $\varphi \in \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ the image $R(\varphi)$ is an element of $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$. To do so we write $\varphi(x) = x + f(x)$, with $f \in \mathcal{A}_1$. Using a Taylor expansion of $\sqrt{1+x}$ around $x = 0$,

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4}\int_0^1 \frac{1-t}{(1+tx)^{\frac{3}{2}}} dt \, x^2 ,$$

we obtain

$$\begin{aligned} R(\varphi) &= 2\big((\varphi')^{1/2} - 1\big) = 2\sqrt{1+f'} - 2 \\ &= f' - \frac{1}{2}\int_0^1 \frac{1-t}{(1+tf')^{\frac{3}{2}}} f' \, dt \, f' \\ &=: f' + F(f')f' , \end{aligned}$$

with $F \in C^\omega(\mathbb{R}_{>-1}, \mathbb{R})$ satisfying $F(0) = 0$.

Because $\varphi = \mathrm{Id} + f$ is a diffeomorphism, we have $f' > -1$ and since $f' \in \mathcal{A}(\mathbb{R})$ implies that f' vanishes at $-\infty$ and at ∞ we can can even conclude that $f' > -1 + \varepsilon$

for some $\varepsilon > 0$. Therefore $F(f')$ is a bounded function for each $f' \in \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$. Using that all the spaces $\mathcal{A}(\mathbb{R})$ are $\mathcal{B}(\mathbb{R})$-modules we conclude that $F(f')f'$ and hence $R(\varphi)$ are elements of $\mathcal{A}(\mathbb{R})$.

To check that the mapping $R : \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}) \to \mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big)$ is bijective, we use the identity $\frac{1}{4}(\gamma(x)+2)^2 - 1 = f'(x)$ with $\gamma = R(\varphi) = R(\mathrm{Id}+f)$. Using this it is straightforward to calculate that

$$R \circ R^{-1} = \mathrm{Id}_{\mathcal{A}}, \qquad R^{-1} \circ R = \mathrm{Id}_{\mathrm{Diff}} \ .$$

To compute the pullback of the L^2-metric via the R-map we first need to calculate its tangend mapping. For this let $h = X \circ \varphi \in T_\varphi \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ and let $t \mapsto \psi(t)$ be a smooth curve in $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ with $\psi(0) = \mathrm{Id}$ and $\partial_t|_0 \psi(t) = X$. We have:

$$\begin{aligned}
T_\varphi R.h &= \partial_t|_0 R(\psi(t)\circ\varphi) = \partial_t|_0 2\Big(((\psi(t)\circ\varphi)_x)^{1/2} - 1\Big) = \partial_t|_0 2((\psi(t)_x \circ \varphi)\,\varphi_x)^{1/2} \\
&= 2(\varphi_x)^{1/2}\partial_t|_0((\psi(t)_x)^{1/2}\circ\varphi) = (\varphi_x)^{1/2}\Big(\frac{\psi_{tx}(0)}{(\psi(0)_x)^{-1/2}} \circ \varphi\Big) \\
&= (\varphi_x)^{1/2}(X' \circ \varphi) = (\varphi')^{1/2}(X' \circ \varphi) \ .
\end{aligned}$$

Using this formula we have for $h = X_1 \circ \varphi, k = X_2 \circ \varphi$:

$$R^*\langle h, k\rangle_{L^2} = \langle T_\varphi R.h, T_\varphi R.k\rangle_{L^2} = \int_{\mathbb{R}} X_1'(x) X_2'(x)\, dx = \langle h, k\rangle_{\dot{H}^1} \ . \qquad \square$$

4.4. **Corollary.** *Given $\varphi_0, \varphi_1 \in \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ the geodesic $\varphi(t,x)$ connecting them is given by*

$$\varphi(t,x) = R^{-1}\Big((1-t)R(\varphi_0) + tR(\varphi_1)\Big)(x) \tag{11}$$

and their geodesic distance is

$$d(\varphi_0,\varphi_1)^2 = 4\int_{\mathbb{R}} \big((\varphi_1')^{1/2} - (\varphi_0')^{1/2}\big)^2\, dx \ . \tag{12}$$

Furthermore the support of the geodesic is localized in the following sense: if $\varphi(t,x) = x + f(t,x)$ with $f(t) \in \mathcal{A}_1(\mathbb{R})$ and similarly for φ_0, φ_1, then $\mathrm{supp}(\partial_x f(t))$ *is contained in* $\mathrm{supp}(\partial_x f_0) \cup \mathrm{supp}(\partial_x f_1)$

Proof. The formula for the geodesic $\varphi(t,x)$ is clear. The geodesic distance between φ_0, φ_1 is given as the L^2 difference between their R-maps:

$$\begin{aligned}
d^{\mathrm{Diff}}(\varphi_0,\varphi_1) = d^{\mathcal{A}}(R(\varphi_0), R(\varphi_1)) &= \int_0^1 \sqrt{\int_{\mathbb{R}} \big(R(\varphi_1) - R(\varphi_0)\big)^2\, dx}\; dt \\
&= 2\sqrt{\int_{\mathbb{R}} \big((\varphi_1')^{1/2} - (\varphi_0')^{1/2}\big)^2\, dx} \ .
\end{aligned}$$

To prove the statement regarding the support of the geodesic we use the inversion formula of R to obtain

$$f'(t,x) = \varphi'(t,x) - 1 = \frac{1}{4}\gamma(t,x)(\gamma(t,x)+4) \ ,$$

where $\gamma(t,x) = (1-t)R(\varphi_0)(x) + tR(\varphi_1)(x)$ is the image of the geodesic under the R-map. Next we note that at the points, where $f_i'(x) = 0$ we have $\varphi_i'(x) = 1$ and $R(\varphi_i)(x) = 0$. Hence at the points where both $f_0'(x) = f_1'(x) = 0$ we also have $f'(t,x) = 0$ for all $t \in [0,1]$. $\square$

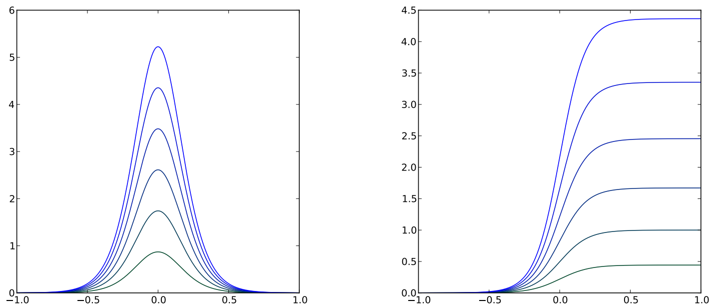

FIGURE 1. A complete geodesic $\varphi(t, x) \in \operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R})$ sampled at time points $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$. Left image: The geodesic in the R-map space. Right image: The geodesic in the original space, visualized as $\varphi(t, x) - x$.

Example. A geodesic connecting the identity to the diffeomorphism $\varphi_1(x) = x + e^{-1/(x+1)^2} e^{-1/(x-1)^2}$ can be seen in Figure 1. In all the examples presented in this article, we consider diffeomorphisms φ with $\operatorname{supp}(\varphi' - 1) \subset [-1, 1]$. We approximated the diffeomorphisms with 1000 points on this interval. In the following Lemma it is shown that this behavior does not hold in general:

4.5. **Lemma.** *The metric space $\left(\operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1\right)$ is path-connected and geodesically convex but not geodesically complete.*

In particular, for every $\varphi_0 \in \operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R})$ and $h \in T_{\varphi_0} \operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R})$, $h \neq 0$ there exists a time $T \in \mathbb{R}$ such that $\varphi(t, \cdot)$ is a geodesic for $|t| < |T|$ starting at φ_0 with $\varphi_t(0) = h$, but $\varphi_x(T, x) = 0$ for some $x \in \mathbb{R}$.

Proof. Set $\gamma_0 = R(\varphi_0$ and $k = T_{\varphi_0} R.h$. Then the R-mapped form $\gamma(t) = R(\varphi(t))$ of the geodesic $\varphi(t)$ is

$$\gamma(t, x) = \gamma_0(x) + tk(x)$$

and the geodesic ceases to exist, when we leave the image of the R-map, i.e. when $\gamma(t, x) = -2$ for some pair (t, x). Consider the function $g(x) = |2 + \gamma_0(x)|/|k(x)|$ and set $g(x) = \infty$ where $k(x) = 0$. As we assumed $h \neq 0$, there is at least one $x \in \mathbb{R}$, such that $g(x)$ is finite. Since $k(x) \to 0$ as $|x| \to \infty$ we have $g(x) \to \infty$ for x large and g attains the minimum at some point. Let this point be x_0 and define $T = -(2 + \gamma_0(x_0))/k(x_0)$. Then $|T|$ is the time, when $\gamma(t, x)$ first reaches -2. So for $|t| < |T|$ the geodesic $\gamma(t)$ lies in $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ and $\gamma(T, x_0) = -2$. Then we have

$$\varphi_x(1, 0) = 1 + \frac{1}{4}\gamma(x)(\gamma(x) + 4)\big|_{x=0} = 1 - 1 = 0\ ,$$

as required. This proves that the space is not geodesically complete.

The statement regarding path-connectedness and geodesically convexity are direct consequences of the path-connectedness and convexity of $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$. $\square$

Example. This behavior is illustrated in Figure 2. There we have again chosen $\varphi_0 = \mathrm{Id}$ and we have solved the geodesic equation in direction

$$h(x) = R^{-1}\left(\frac{-1}{4 + 4e^{-10x}}\right)$$

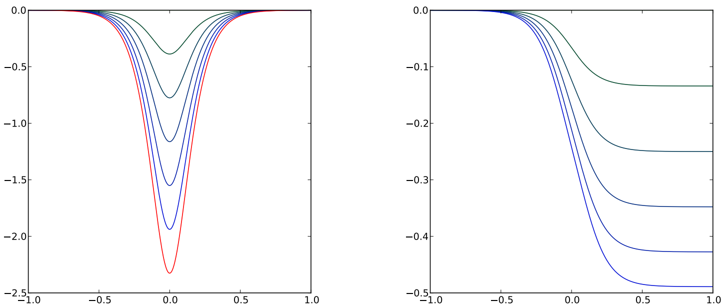

FIGURE 2. An incomplete geodesic $\varphi(t,x) \in \operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R})$. Left image: The geodesic in the R-map space at time points $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$. At time $t = 3$ (red line) the geodesic has already left the space of R-maps. Right image: The geodesic in the original space visualized as $\varphi(t,x) - x$ sampled at time points $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$.

until the geodesic leaves the space of diffeomorphisms – which happens approximately at time $t = 2.58$.

4.6. **The square root representation on** $\operatorname{Diff}_{\mathcal{A}}(\mathbb{R})$**.** We will now study the homogeneous H^1 metric on diffeomorphism groups that do not allow a shift towards infinity. We can still use the same square root representation as in the previous section, but now the image of this map will be a splitting submanifold in the image space $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$.

Theorem. *The square root representation on the diffeomorphism group* $\operatorname{Diff}_{\mathcal{A}}(\mathbb{R})$ *is a bijective mapping, given by:*

$$R : \begin{cases} \operatorname{Diff}_{\mathcal{A}}(\mathbb{R}) \to \big(\operatorname{Im}(R), \|\cdot\|_{L^2}\big) \subset \big(\mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big), \|\cdot\|_{L^2}\big) \\ \qquad\qquad \varphi \mapsto 2\left((\varphi')^{1/2} - 1\right). \end{cases}$$

The pull-back of the restriction of the flat L^2-metric to $\operatorname{Im}(R)$ *via R is again the homogeneous Sobolev metric of order one. The image of the R-map is the splitting submanifold (in the sense of* [18, Sect. 27.11]*) of* $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ *given by:*

$$\operatorname{Im}(R) = \left\{ \gamma \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2}) : F(\gamma) := \int_{\mathbb{R}} \gamma\big(\gamma + 4\big)\, dx = 0 \right\}.$$

Proof. The statement regarding the image of R, follows from the fact that a diffeomorphism $\varphi = \operatorname{Id} + g \in \operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R})$ is also an element of $\operatorname{Diff}_{\mathcal{A}}(R)$ iff

$$\int_{\mathbb{R}} (\varphi'(x) - 1)\, dx = \int_{\mathbb{R}} g'(x)\, dx = 0\,.$$

Using that for $\gamma = R(\varphi)$ we have $\frac{1}{4}(\gamma(x) + 2)^2 - 1 = g'(x)$ we obtain the desired result. The mapping $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2}) \ni f \mapsto 2((f+1)^{1/2} - 1) \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ is a diffeomorphism and it maps $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-1}) \cap \{f : \int f\, dx = 0\}$ diffeomorphically onto $\operatorname{Im}(R)$, which therefore is a splitting submanifold. $\square$

Remark. Note that we have $dF(\gamma)(\delta) = \int_\mathbb{R}(2\gamma+4).\delta\, dx$, but $2\gamma+4$ is not in $\mathcal{A}$, only in the dual $\mathcal{A}^*$. So the 'normal field' along the codimension 1 submanifold $\mathrm{Im}(\mathbb{R})$ has no length; methods like the Gauss formula or the Gauss equation do not make sense. This is a diffeomorphic translation of the nonexistence of the geodesic equation in $\mathrm{Diff}_\mathcal{A}(\mathbb{R})$.

4.7. **Geodesic distance on** $\mathrm{Diff}_\mathcal{A}(\mathbb{R})$**.** We have seen that on the space $\mathrm{Diff}_\mathcal{A}(\mathbb{R})$ the geodesic equation does not exist. It is however possible to define the geodesic distance between two diffeomorphisms $\varphi_0, \varphi_1 \in \mathrm{Diff}_\mathcal{A}(\mathbb{R})$ as the infimum over all paths $\varphi : [0,1] \to \mathrm{Diff}_\mathcal{A}(\mathbb{R})$ connecting these,

$$d^\mathcal{A}(\varphi_0,\varphi_1) = \inf_{\substack{\varphi(0)=\varphi_0\\ \varphi(1)=\varphi_1}} \int_0^1 \sqrt{G_{\varphi(t)}(\partial_t\varphi(t), \partial_t\varphi(t))}\, dt\, .$$

In Corollary 4.4 we gave an explicit formula for the geodesic distance $d^{\mathcal{A}_1}$ on the space $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$. It turns out that the geodesic distance $d^\mathcal{A}$ on $\mathrm{Diff}_\mathcal{A}(\mathbb{R})$ is the same as the restriction of $d^{\mathcal{A}_1}$ to $\mathrm{Diff}_\mathcal{A}(\mathbb{R})$. Note that this is not a trivial statement, since in the definition of $d^\mathcal{A}$ the infimum is taken over all paths lying in the smaller space $\mathrm{Diff}_\mathcal{A}(\mathbb{R})$.

Theorem. *The geodesic distance* $d^\mathcal{A}$ *on* $\mathrm{Diff}_\mathcal{A}(\mathbb{R})$ *coincides with the restriction of* $d^{\mathcal{A}_1}$ *to* $\mathrm{Diff}_\mathcal{A}(\mathbb{R})$*, i.e. for* $\varphi_0,\varphi_1 \in \mathrm{Diff}_\mathcal{A}(\mathbb{R})$ *we have*

$$d^\mathcal{A}(\varphi_0,\varphi_1) = d^{\mathcal{A}_1}(\varphi_0,\varphi_1)\, .$$

Proof. Because the space $\mathrm{Diff}_\mathcal{A}(\mathbb{R})$ is smaller than $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$, we have the inequality $d_{\mathcal{A}_1} \le d_\mathcal{A}$. The argument below will establish the other inequality $d_\mathcal{A} \le d_{\mathcal{A}_1}$ and hence together we will be able to conclude that $d_{\mathcal{A}_1} = d_\mathcal{A}$.

Take $\varphi_0,\varphi_1 \in \mathrm{Diff}_\mathcal{A}(\mathbb{R})$ and let $\varphi(t)$ be a path connecting them in the larger space $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$. Then $\gamma(t) = R(\varphi(t)$ is a path in $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ and its length is measured by

$$L(\gamma) = \int_0^1 \sqrt{\int_\mathbb{R} \partial_t\gamma(t,x)^2 dx} dt\, .$$

We also have the functional

$$F(\gamma) = \int_\mathbb{R} \gamma(x)^2 + 4\gamma(x) dx\, ,$$

which measures whether $\gamma \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ lies in the image of $\mathrm{Diff}_\mathcal{A}(\mathbb{R})$ under the R-map.

We will construct a sequence $\widetilde{\gamma}_n$ of paths with $n \to \infty$, such that these paths satisfy $F(\widetilde{\gamma}_n) = 0$ and $L(\widetilde{\gamma}_n) \to L(\widetilde{\gamma})$. This will show that each path in $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ can be approximated arbitrary well by path in the smaller space $\mathrm{Diff}_\mathcal{A}(\mathbb{R})$ and hence we will have established the other inequality $d_\mathcal{A} \le d_{\mathcal{A}_1}$.

Since $\gamma(t)$ and $\partial_t\gamma(t)$ decay to 0 as $x \to \infty$ there exists for each $n > 0$ some $x_n \in \mathbb{R}$ such that

$$\left.\begin{aligned} |\gamma(t,x)| &< \tfrac{1}{n} \\ |\partial_t\gamma(t,x)| &< \tfrac{1}{n} \end{aligned}\right. \qquad \text{for } x > x_n\, .$$

We also define $\varepsilon_n = \frac{1}{n}$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function with support in $[-1,1]$, $\psi(x) \ge 0$ and with $\int \psi(x) dx = 1$ and $\int \psi(x)^2 dx = 1$. Now define the new path

$$\widetilde{\gamma}_n(t,x) = \gamma(t,x) + \alpha_n(t)\varepsilon_n\psi\left(\varepsilon_n(x - x_n - \varepsilon_n^{-1})\right)$$

with the function $\alpha_n(t)$ determined by $F(\widetilde{\gamma}_n(t)) = 0$. To make calculations easier we set $\psi_n(x) = \psi\left(\varepsilon_n(x - x_n - \varepsilon_n^{-1})\right)$ and we note that

$$\varepsilon_n \int_{\mathbb{R}} \psi_n(x) dx = 1\,, \qquad\qquad \varepsilon_n \int_{\mathbb{R}} \psi_n(x)^2 dx = 1\,.$$

Writing the condition $F(\widetilde{\gamma}_n(t)) = 0$ more explicitly we get

$$\begin{aligned} F(\widetilde{\gamma}_n(t)) &= \int_{\mathbb{R}} \gamma(t)^2 + 2\alpha_n(t)\varepsilon_n\psi_n\gamma(t) + \alpha_n(t)^2\varepsilon_n^2\psi_n^2 + 4\left(\gamma(t) + \alpha_n(t)\varepsilon_n\psi_n\right) dx \\ &= \varepsilon_n\alpha_n(t)^2 + \left(4 + 2\varepsilon_n \int_{\mathbb{R}} \psi_n\gamma(t)dx\right)\alpha_n(t) + F(\gamma(t))\,. \end{aligned}$$

We need to estimate the integral

$$C_n(t) := \varepsilon_n \int_{\mathbb{R}} \psi_n(x)\gamma(t,x)dx$$

to be able to control $\alpha(t)$. We have

$$\begin{aligned} |C_n(t)| &= \left|\varepsilon_n \int_{\mathbb{R}} \psi\left(\varepsilon_n(x - x_n - \varepsilon_n^{-1})\right)\gamma(t,x)dx\right| \\ &\leq \varepsilon_n \int_{x_n}^{x_n + 2\varepsilon_n^{-1}} \psi\left(\varepsilon_n(x - x_n - \varepsilon_n^{-1})\right)|\gamma(t,x)|\, dx \\ &\leq \varepsilon_n \int_{x_n}^{x_n + 2\varepsilon_n^{-1}} \psi\left(\varepsilon_n(x - x_n - \varepsilon_n^{-1})\right)\frac{1}{n} dx \\ &\leq \frac{1}{n}\,. \end{aligned}$$

Hence we see that for large n we will have $\alpha_n(t) \to -F(\gamma(t))/4$ and the convergence is uniform in t. To see that $\alpha(t)$ is smooth in t we can use the explicit formula

$$\alpha_n(t) = -\varepsilon_n^{-1}\left(2 + C_n(t)\right) - \sqrt{\varepsilon_n^{-2}\left(2 + C_n(t)\right)^2 - \varepsilon_n^{-1}F(\gamma(t))}$$

and note that for ε_n sufficiently small the term under the square-root will always be positive.

Now we look at the length of the path $\widetilde{\gamma}_n$. Let us consider only the inner integral $\int_{\mathbb{R}} \partial_t\widetilde{\gamma}_n(t,x)dx$. We will show the convergence

$$\int_{\mathbb{R}} \left(\partial_t\widetilde{\gamma}_n(t,x)\right)^2 dx \to \int_{\mathbb{R}} \left(\partial_t\gamma(t,x)\right)^2 dx$$

uniformly in t, which will imply the convergence $L(\widetilde{\gamma}_n) \to L(\gamma)$. Thus we look at

$$\begin{aligned} \int_{\mathbb{R}} \left(\partial_t\widetilde{\gamma}_n(t)\right)^2 dx &= \int_{\mathbb{R}} \left(\partial_t\gamma(t)\right)^2 + 2\partial_t\alpha_n(t)\varepsilon_n\psi_n\partial_t\gamma(t) + \left(\partial_t\alpha_n(t)\right)^2\varepsilon_n^2\psi_n(x)^2 dx \\ &= \int_{\mathbb{R}} \left(\partial_t\gamma(t)\right)^2 dx + 2\partial_t\alpha_n(t)\varepsilon_n \int_{\mathbb{R}} \psi_n\partial_t\gamma(t)dx + \left(\partial_t\alpha_n(t)\right)^2\varepsilon_n\,. \end{aligned}$$

The integral

$$D_n(t) := \varepsilon_n \int_{\mathbb{R}} \psi_n(x)\partial_t\gamma(t,x)dx$$

can be estimated in the same way as $C_n(t)$ before to obtain $D_n(t) \leq \frac{1}{n}$. It remains to bound $\partial_t\alpha(t)$ uniformly in t to show convergence. For this end we differentiate the equation

$$\varepsilon_n\alpha_n(t)^2 + \left(4 + 2\varepsilon_n \int_{\mathbb{R}} \psi_n\gamma(t)dx\right)\alpha_n(t) + F(\gamma(t)) = 0\,,$$

that defines $\alpha(t)$, which gives us

$$\partial_t\alpha_n(t)\left(2\varepsilon_n\alpha_n(t)+4+2C_n(t)\right)+T_{\gamma(t)}F.\partial_t\gamma(t)=0$$

and thus

$$\partial_t\alpha_n(t)=-\frac{T_{\gamma(t)}F.\partial_t\gamma(t)}{4+2\varepsilon_n\alpha_n(t)+2C_n(t)}\ .$$

We see that we have $\partial_t\alpha_n(t)\to -T_{\gamma(t)}F.\partial_t\gamma(t)/4$ and the convergence is uniform in t. Thus we have shown that the convergence of the length functional $L(\widetilde{\gamma}_n)\to L(\gamma)$. $\square$

4.8. **The submanifold** $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ **inside** $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$**.** The following theorem deals with the question, how $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ lies inside the extension $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$. We give an upper bound for how often a geodesic in $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ might intersect or be tangent to $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$. It is only an upper bound, because the geodesic might leave the group of diffeomorphisms, before intersecting $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$.

Theorem. *Consider a geodesic $\varphi(t)$ in $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ starting at $\varphi(0)=\varphi_0$ with initial velocity $\varphi_t\varphi(0)=u_0\circ\varphi_0$ and denote by $u(t)=\partial_t\varphi(t)\circ\varphi(t)^{-1}$ the right-trivialized velocity. Then the size of the shift at infinity is given by*

$$\mathrm{Shift}(\varphi(t))=\mathrm{Shift}(\varphi_0)+tu_0(\infty)+\frac{t^2}{4}\int_{\mathbb{R}}(u_0')^2dx$$
$$u(t,\infty)=u_0(\infty)+t\int_{\mathbb{R}}(u_0')^2dx\ .$$

This means that every geodesic in $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ intersects $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ at most twice and every geodesic is tangent to a right-coset of $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ at most once.

For $\varphi_0,\varphi_1\in\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ we can give the following formula for the size of the shift

$$\mathrm{Shift}(\varphi(t))=\frac{t^2-t}{4}\left\|R(\varphi_0)-R(\varphi_1)\right\|^2_{L^2}=(t^2-t)\left\|\sqrt{\varphi_0'}-\sqrt{\varphi_1'}\,\right\|^2_{L^2}\ .$$

Proof. To make computations easier define the following variables

$$\begin{aligned}\gamma_0&=R(\varphi_0)\\ k_0&=T_{\varphi_0}R.(u_0\circ\varphi_0)=\sqrt{\varphi_0'}u_0'\circ\varphi_0\\ \gamma(t)&=R(\varphi(t))=\gamma_0+tk_0\ .\end{aligned}$$

For a diffeomorphism $\varphi\in\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ the size of the shift at ∞ is given by

$$\mathrm{Shift}(\varphi)=\lim_{x\to\infty}\varphi(x)-x=\int_{-\infty}^{\infty}\varphi'(x)-1\,dx=\frac{1}{4}\int_{-\infty}^{\infty}R(\varphi)^2+4R(\varphi)\,dx\ .$$

Similarly the value of a function $u\in\mathcal{A}_1$ at ∞ can be computed by

$$\begin{aligned}u(\infty)&=\int_{-\infty}^{\infty}u'(x)dx=\int_{-\infty}^{\infty}(u'\circ\varphi)(x)\varphi'(x)\,dx\\ &=\frac{1}{2}\int_{-\infty}^{\infty}(R(\varphi)+2)T_\varphi R.(u\circ\varphi)(x)\,dx\ .\end{aligned}$$

Note that this holds for any $\varphi\in\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$. Since the R-map pulls back the L^2-metric to the $\dot{H}^1$-metric we also have the identity

$$\int_{-\infty}^{\infty}u'(x)dx=\int_{-\infty}^{\infty}(T_\varphi R.(u\circ\varphi))^2dx\ .$$

For the sake of convenience let us rewrite the last two using the variables u_0, γ_0 and k_0.

$$u_0(\infty) = \frac{1}{2}\int_{-\infty}^{\infty} (\gamma_0 + 2)k_0 \, dx \qquad\qquad \int_{-\infty}^{\infty} (u_0')^2 \, dx = \int_{-\infty}^{\infty} k_0^2 \, dx$$

Now computing the shift of $\varphi(t)$ at ∞ is easy,

$$\begin{aligned}\mathrm{Shift}(\varphi(t)) &= \frac{1}{4}\int_{-\infty}^{\infty} (\gamma_0 + tk_0)^2 + 4(\gamma_0 + tk_0) \, dx \\ &= \frac{1}{4}\int_{-\infty}^{\infty} \gamma_0^2 + 4\gamma_0 + 2t\gamma_0 k_0 + 4tk_0 + t^2 k_0^2 \, dx \\ &= \mathrm{Shift}(\varphi_0) + tu_0(\infty) + \frac{t^2}{4}\int_{-\infty}^{\infty} (u_0')^2 dx \, .\end{aligned}$$

Computing the value of $u(t)$ at ∞ is just as simple,

$$\begin{aligned}u(t,\infty) &= \frac{1}{2}\int_{-\infty}^{\infty} (\gamma_0 + tk_0 + 2)k_0 \, dx \\ &= u_0(\infty) + t\int_{\mathbb{R}} (u_0')^2 dx \, .\end{aligned}$$

If we start with $\varphi_0, \varphi_1 \in \mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$, then the geodesic connecting them has $k_0 = \gamma_1 - \gamma_0$ with $\gamma_1 = R(\varphi_1)$. Some algebraic manipulations, keeping in mind that $\mathrm{Shift}(\varphi_i) = \frac{1}{4}\int \gamma_i^2 + 4\gamma_i = 0$, give us

$$\begin{aligned}u_0(\infty) &= \frac{1}{2}\int_{-\infty}^{\infty} (\gamma_0 + 2)(\gamma_1 - \gamma_0) \, dx = \frac{1}{2}\int_{-\infty}^{\infty} \gamma_0\gamma_1 - \gamma_0^2 - 2\gamma_0 + 2\gamma_1 \, dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \gamma_0\gamma_1 - \gamma_0^2 + \frac{1}{2}\gamma_0^2 - \frac{1}{2}\gamma_1^2 \, dx = -\frac{1}{4}\int_{-\infty}^{\infty} (\gamma_1 - \gamma_0)^2 \, dx \, ,\end{aligned}$$

which in turn leads to

$$\mathrm{Shift}(\varphi(t)) = tu_0(\infty) + \frac{t^2}{4}\int_{-\infty}^{\infty} (u_0')^2 \, dx = \frac{t^2 - t}{4}\int_{-\infty}^{\infty} (\gamma_1 - \gamma_0)^2 \, dx \, .$$

This completes the proof. □

An example of a geodesic illustrating the behavior described in the lemma can be seen in Figure 3.

4.9. **Solving the Hunter–Saxton equation.** The theory described in the previous sections allows us to construct an analytic solution formula for the Hunter–Saxton equation on $\mathcal{A}_1(\mathbb{R})$. Here $\mathcal{A}_1(\mathbb{R})$ denotes one of the function spaces $C_{c,1}^{\infty}(\mathbb{R})$, $\mathcal{S}_1(\mathbb{R})$ or $H_1^{\infty}(\mathbb{R})$, as defined in 2.2, 2.3, and 2.4.

Theorem (Solutions to the HS–equation)**.** *Given an initial value u_0 in $\mathcal{A}_1(\mathbb{R})$ the solution to the Hunter–Saxton equation is given by:*

$$u(t,x) = \varphi_t(t, \varphi^{-1}(t,x)), \quad \textit{with} \quad \varphi(t,x) = R^{-1}\big(tu_0'\big)(x) \, .$$

In particular this means that a solution with initial condition in one of the spaces $C_c^{\infty}(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$ or $H^{\infty}(\mathbb{R})$ exists for all time $t > 0$, if and only if $u_0'(x) \geq 0$ for all $x \in \mathbb{R}$. All solutions are real analytic in time in the sense of [20, Section 9].

Proof. By the theory of the previous sections we know that the path

$$\varphi(t,x) = R^{-1}(t\gamma)(x)$$

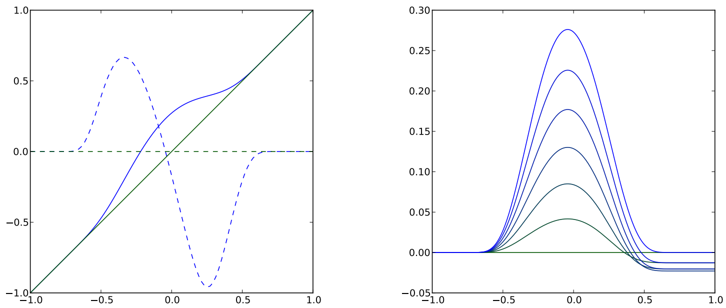

FIGURE 3. Geodesic $\varphi(t,x) \in \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ between two diffeomorphisms in $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ sampled at times $t = 0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 1$. Left image: The geodesic in the R-map space. Right image: The geodesic in the original space, visualized as $\varphi(t,x) - x$.

with $\varphi(0,x) = x$ is a solution to the geodesic equation for every $\gamma \in \mathcal{A}_1(\mathbb{R})$. It remains to choose γ such that the initial condition

$$\varphi_t(0, \varphi^{-1}(0,x)) = \varphi_t(0,x) = u_0(x)$$

is satisfied. This can be achieved by choosing $\gamma = T_{\mathrm{Id}}R.u_0$, since

$$\gamma = \partial_t|_0(t\gamma) = \partial_t|_0 R(\varphi(t)) = T_{\varphi(t)}R.\varphi_t(t)\big|_0 = T_{\mathrm{Id}}R.u_0\ .$$

Using the formula $T_\varphi R.h = (\varphi')^{-1/2}h'$ from the proof of Theorem 4.3 yields $T_{\mathrm{Id}}R.u_0 = u_0'$.

The solution is real analytic in t since $t \mapsto t.u_0$ is a real analytic curve in $\mathcal{A}_1(\mathbb{R})$ and since R^{-1} respects real analytic curves; see [20, Section 9].

Given x_0, u_0 such that $u_0'(x_0) < 0$ there exist $t_0 \in \mathbb{R}$ with $t_0 u_0'(x_0) < -2$. Thus the geodesic at time t_0 has left the R-map space and the solution of the Hunter-Saxton equation leaves the space $\mathcal{A}_1$. □

A more explicit formula for the solution is given by

$$\begin{aligned} u(t,x) &= u_0(\varphi^{-1}(t,x)) + \frac{t}{2}\int_{-\infty}^{\varphi^{-1}(t,x)} u_0'(y)^2\, dy \\ \varphi(t,x) &= x + \frac{1}{4}\int_{-\infty}^{x} t^2(u_0'(y))^2 + 4tu_0'(y)\, dy\ . \end{aligned}$$

Remark. The Hunter-Saxton equation on the real line also provides an example to see how geometry and PDE behaviour influence each other. It was shown in 4.2 that the geodesic equation on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ does not exist, because the condition $\mathrm{ad}^*(u)\,u \in \check{G}_{\mathrm{Id}}(u)$ was not satisfied. From a naive point of view we could start with the energy

$$E(u) = \int_0^1 \int_{-\infty}^{\infty} u_x^2\, dx\, dt \tag{13}$$

defined on functions $u \in C^\infty([0,1], \mathcal{A}(\mathbb{R}))$ and take variations of the form $\delta u = \eta_t + \eta_x u - \eta u_x$ with fixed endpoints $\eta(0,x) = \varphi_0(x)$ and $\eta(1,x) = \varphi_1(x)$. This

would lead, after some partial integration, to

$$\langle DE(u), \delta u\rangle = \int_0^1 \int_{-\infty}^{\infty} (u_{txx} + (uu_x)_{xx} - u_x u_{xx})\, \eta \, dx\, dt$$

and we could now declare

$$u_{txx} + (uu_x)_{xx} - u_x u_{xx} = 0$$

to be the "geodesic equation". It is any case the equation, that critical points of the energy functional (13) have to satisfy. But this equation has no solutions in $\mathcal{A}$. It is shown in Theorem 4.8 that a solution $u \in C^\infty([0,1], \mathcal{A}_1(\mathbb{R}))$ meets $\mathcal{A}(\mathbb{R})$ at most once. In order to find solutions we have to enlarge the space to $\mathcal{A}_1(\mathbb{R})$ and then the above Theorem via the R-map gives us existence of solutions.

4.10. **Continuing geodesics beyond the group, or how do solutions of the Hunter-Saxton equation blow up.** Consider a straight line $\gamma(t) = \gamma_0 + t\gamma_1$ in $\mathcal{A}(\mathbb{R},\mathbb{R})$. Then $\gamma(t) \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ precisely for t in an open interval (t_0, t_1) which is finite at least on one side; at $t_1 < \infty$, say. Note that

$$\varphi(t)(x) := R^{-1}(\gamma(t))(x) = x + \frac{1}{4}\int_{-\infty}^{x} \gamma^2(t)(u) + 4\gamma(t)(u)\, du$$

makes sense for all t, and that $\varphi(t) : \mathbb{R} \to \mathbb{R}$ is smooth and $\varphi(t)'(x) \geq 0$ for all x and t so that $\varphi(t)$ is monotone non-decreasing. Moreover, $\varphi(t)$ is proper and surjective, since $\gamma(t)$ vanishes at $-\infty$ and ∞. Let

$$\operatorname{Mon}_{\mathcal{A}_1}(\mathbb{R}) := \big\{ \operatorname{Id} + f : f \in \mathcal{A}_1(\mathbb{R},\mathbb{R}), f' \geq -1 \big\}$$

be the monoid (under composition) of all such functions.

For $\gamma \in \mathcal{A}(\mathbb{R},\mathbb{R})$ let $x(\gamma) := \min\{x \in \mathbb{R} \cup \{\infty\} : \gamma(x) = -2\}$. Then for the line $\gamma(t)$ from above we see that $x(\gamma(t)) < \infty$ for all $t > t_1$. Thus, if the geodesic $\varphi(t)$ leaves the diffeomorphism group at t_1, it never comes back but stays inside $\operatorname{Mon}_{\mathcal{A}_1}(\mathbb{R})$ for the rest of its life. In this sense $\operatorname{Mon}_{\mathcal{A}_1}(\mathbb{R})$ is a *geodesic completion* of $\operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R})$, and $\operatorname{Mon}_{\mathcal{A}_1}(\mathbb{R}) \setminus \operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R})$ is the *boundary.*

What happens to the corresponding solution $u(t,x) = \varphi_t(t, \varphi(t)^{-1}(x))$ of the Hunter-Saxton equation: in certain points it has infinite derivative, it may be multivalued, or its graph can contain whole vertical intervals. If we replace an element $\varphi \in \operatorname{Mon}_{\mathcal{A}_1}(\mathbb{R})$ by its graph $\{(x, \varphi(x)) : x \in \mathbb{R}\} \subset \mathbb{R}$ we get a smooth 'monotone' submanifold, a smooth monotone relation. The inverse φ^{-1} is then also a smooth monotone relation. Then $t \mapsto \{(x, u(t,x)) : x \in \mathbb{R}\}$ is a (smooth) curve of relations. To check that it satisfies the Hunter-Saxton equation is an exercise for the interested reader. We have described the *flow completion* of the Hunter-Saxton equation in the spirit of [15].

4.11. **Soliton-like solutions.** For a right invariant metric on a diffeomorphism group one can ask whether (generalized) solutions $u(t) = \varphi_t(t) \circ \varphi(t)^{-1}$ exist such that the momenta $\check{G}(u(t)) =: p(t)$ are distributions with finite support. Here the geodesic $\varphi(t)$ may exist only in some suitable Sobolev-completion of the diffeomorphism group. By the general theory (see in particular [33, 4.4], or [37]) the momentum $\operatorname{Ad}(\varphi(t))^* p(t) = \varphi(t)^* p(t) = p(0)$ is constant. In other words, $p(t) = (\varphi(t)^{-1})^* p(0) = \varphi(t)_* p(0)$, i.e., the momentum is carried forward by the flow and remains in the space of distributions with finite support. The infinitesimal version (take ∂_t of the last expression) is $p_t(t) = -\mathcal{L}_{u(t)} p(t) = -\operatorname{ad}_{u(t)}{}^* p(t)$; compare with 3.2. The space of N-solitons of order 0 consists of momenta of the

form $p_{y,a} = \sum_{i=1}^N a_i \delta_{y_i}$ with $(y,a) \in \mathbb{R}^{2N}$. Consider an initial soliton $p_0 = \check{G}(u_0) = -u_0'' = \sum_{i=1}^N a_i\, \delta_{y_i}$ with $y_1 < y_2 < \cdots < y_N$. Let H be the Heaviside function

$$H(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases},$$

and $D(x) = 0$ for $x \leq 0$ and $D(x) = x$ for $x > 0$. We will see later, why the choice $H(0) = \frac{1}{2}$ is the most natural one – note that behavior is called the Gibbs phenomenon. With these functions we can write

$$\begin{aligned} u_0''(x) &= -\sum_{i=1}^N a_i \delta_{y_i}(x) \\ u_0'(x) &= -\sum_{i=1}^N a_i H(x - y_i) \\ u_0(x) &= -\sum_{i=1}^N a_i D(x - y_i)\ . \end{aligned}$$

We will assume from now on that $\sum_{i=1}^N a_i = 0$. Then $u_0(x)$ is constant for $x > y_N$ and thus $u_0 \in H_1^1(\mathbb{R})$; with slight abuse of notation we assume that $H_1^1(\mathbb{R})$ is defined similarly as $H_1^\infty(\mathbb{R})$. Defining $S_i = \sum_{j=1}^i a_j$ we can write

$$u_0'(x) = -\sum_{i=1}^N S_i \left(H(x - y_i) - H(x - y_{i+1}) \right)\ .$$

This formula will be useful, because $\operatorname{supp}(H(.-y_i) - H(.-y_{i+1})) = [y_i, y_{i+1}]$.

The evolution of the geodesic $u(t)$ with initial value $u(0) = u_0$ can be described by a system of ODEs for the variables (y,a). We cite the following result

4.12. **Theorem** ([14]). *The map $(y,a) \mapsto \sum_{i=1}^N a_i \delta_{y_i}$ is a Poisson map between the canonical symplectic structure on $\mathbb{R}^{2N}$ and the Lie-Poisson structure on the dual $T^*_{\mathrm{Id}} \operatorname{Diff}_{\mathcal{A}}(\mathbb{R})$ of the Lie algebra.*

In particular this means that the ODEs for (y,a) are Hamilton's equations for the pull-back Hamiltonian

$$E(y,a) = \frac{1}{2} G_{\mathrm{Id}}(u_{(y,a)}, u_{(y,a)})\ ,$$

with $u_{(y,a)} = \check{G}^{-1}(\sum_{i=1}^N a_i \delta_{y_i}) = -\sum_{i=1}^N a_i D(.-y_i)$. We can obtain the more explicit expression

$$\begin{aligned} E(y,a) &= \frac{1}{2} \int_{\mathbb{R}} \left(u_{(y,a)}(x)' \right)^2 dx = \frac{1}{2} \int_{\mathbb{R}} \left(\sum_{i=1}^N S_i \mathbb{1}_{[y_i, y_{i+1}]} \right)^2 dx \\ &= \frac{1}{2} \sum_{i=1}^N S_i^2 (y_{i+1} - y_i)\ . \end{aligned}$$

Hamilton's equations $\dot{y}_i = \partial E / \partial a_i$, $\dot{a}_i = -\partial E / \partial y_i$ are in this case

$$\begin{aligned} \dot{y}_i(t) &= \sum_{j=i}^{N-1} S_i(t)(y_{i+1}(t) - y_i(t)) \\ \dot{a}_i(t) &= \frac{1}{2} \left(S_i(t)^2 - S_{i-1}(t)^2 \right)\ . \end{aligned}$$

Using the R-map we can find explicit solutions for these equations as follows. Let us write $a_i(0) = a_i$ and $y_i(0) = y_i$. By Theorem 4.9 the geodesic with initial velocity u_0 is given by

$$\begin{aligned}
\varphi(t,x) &= x + \frac{1}{4}\int_{-\infty}^{x} t^2 (u_0'(y))^2 + 4tu_0'(y)\, dy \\
u(t,x) &= u_0(\varphi^{-1}(t,x)) + \frac{t}{2}\int_{-\infty}^{\varphi^{-1}(t,x)} u_0'(y)^2\, dy\ .
\end{aligned}$$

First note that

$$\begin{aligned}
\varphi'(t,x) &= \left(1 + \frac{t}{2}u_0'(x)\right)^2 \\
u'(t,z) &= \frac{u_0'\left(\varphi^{-1}(t,z)\right)}{1 + \frac{t}{2}u_0'\left(\varphi^{-1}(t,z)\right)}\ .
\end{aligned}$$

Using the identity $H(\varphi^{-1}(t,z) - y_i) = H(z - \varphi(t,y_i)$ we obtain

$$u_0'\left(\varphi^{-1}(t,z)\right) = -\sum_{i=1}^{N} a_i H\left(z - \varphi(t,y_i)\right)\ ,$$

and thus

$$\left(u_0'\left(\varphi^{-1}(t,z)\right)\right)' = -\sum_{i=1}^{N} a_i \delta_{\varphi(t,y_i)}(z)\ .$$

Together we get

$$\begin{aligned}
u''(t,z) &= \frac{1}{\left(1 + \frac{t}{2}u_0'\left(\varphi^{-1}(t,z)\right)\right)^2}\left(-\sum_{i=1}^{N} a_i \delta_{\varphi(t,y_i)}(z)\right) \\
&= \sum_{i=1}^{N} \frac{-a_i}{\left(1 + \frac{t}{2}u_0'(y_i)\right)^2}\delta_{\varphi(t,y_i)}(z)\ .
\end{aligned}$$

From here we can read off the solution of Hamilton's equations

$$\begin{aligned}
y_i(t) &= \varphi(t,y_i) \\
a_i(t) &= -a_i\left(1 + \tfrac{t}{2}u_0'(y_i)\right)^{-2}\ .
\end{aligned}$$

When trying to evaluate $u_0'(y_i)$,

$$u_0'(y_i) = a_i H(0) - S_i\ ,$$

we see that u_0' is discontinuous at y_i and it is here, that we seem to have the freedom to choose the value $H(0)$. However it turns out that we observe the Gibbs phenomenon, i.e., only the choice $H(0) = \frac{1}{2}$ leads to solutions of Hamilton's equations. Also the regularized theory of multiplications of distributions (see the discussion in [19, 1.1]) leads to this choice. Thus we obtain:

$$\begin{aligned}
y_i(t) &= y_i + \sum_{j=1}^{i-1}\left(\frac{t^2}{4}S_j^2 - tS_j\right)(y_{j+1} - y_j) \\
a_i(t) &= \frac{-a_i}{\left(1 + \frac{t}{2}\left(\frac{a_i}{2} - S_i\right)\right)^2} = -\left(\frac{S_i}{1 - \frac{t}{2}S_i} - \frac{S_{i-1}}{1 - \frac{t}{2}S_{i-1}}\right)\ .
\end{aligned}$$

It can be checked by direct computation that these functions indeed solve Hamilton's equations.

5. The two-component Hunter-Saxton equation on the real line

In this section we will show, similarly to the previous section, that one can adapt the work of Lenells on the periodic two-component Hunter-Saxton equation [28] to obtain results for the non-periodic case. On the real line this system has been studied from an analytical viewpoint in [41, Section 4].

5.1. **Theorem.** *Let* $\mathcal{M} = \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}) \ltimes \mathcal{A}(\mathbb{R},\mathbb{R})$ *and* $\widetilde{\mathcal{M}} = \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}) \otimes \mathcal{A}(\mathbb{R},\mathbb{R})$ *be the semi-direct product Lie groups with multiplication*

$$(\varphi, \alpha)(\psi, \beta) = (\varphi \circ \psi, \beta + \alpha \circ \psi)\ ,$$

where $\mathcal{A}$ *and* $\mathcal{A}_1$ *are as defined in 4.1. Consider the following weak Riemannian metric* G *on* M *and* $\widetilde{M}$*:*

$$G_{(\mathrm{Id},0)}((X,a),(Y,b)) = \int X'(x)Y'(x) + a(x)b(x)\,dx\ ,$$

where (X,a) *and* (Y,b) *are elements of in the corresponding Lie algebra.*

Then the geodesic equation on $\widetilde{\mathcal{M}}$ *is the 2-component non-periodic Hunter Saxton equation, given by:*

$$\boxed{\begin{aligned} &(u,\rho) = (\varphi_t \circ \varphi^{-1}, \alpha_t \circ \varphi^{-1}) \\ &\qquad u_t = -uu_x + \frac{1}{2}\int_{-\infty}^{x} u_x(z)^2 + \rho^2(z)\,dz \\ &\qquad \rho_t = -(\rho u)_x \end{aligned}}$$

The geodesic equation does not exist on $\mathcal{M}$*, since the adjoint* $\mathrm{ad}((X,a))^* G_{(\mathrm{Id},0)}$ *is not in* $G_{(\mathrm{Id},0)}$*(Lie algebra) for all* (X,a)*. These are not robust Riemannian manifolds in the sense of* [32, 2.4].

Remark. Note that one obtains more "classical" forms of the Hunter-Saxton equation by differentiating the equation for u_t:

$$\begin{aligned} u_{tx} &= -uu_{xx} + \frac{1}{2}(-u_x^2 + \rho^2)\ ; \\ u_{txx} &= \big(-uu_{xx} + \frac{1}{2}(-u_x^2 + \rho^2)\big)_x = -2u_x u_{xx} - uu_{xxx} + \rho\rho_x\ . \end{aligned}$$

The last equation above is the version which Lenells called the 2-component Hunter-Saxton equation in [28].

Proof. The proof the formula for the geodesic equation we need to calculate the adjoint as defined in Section 3.2. For vector fields (X,a) and (Y,b) the Lie-bracket is given by:

$$[(X,a),(Y,b)] = (-[X,Y], -(\mathcal{L}_X b - \mathcal{L}_Y a))\ .$$

We calculate

$$
\begin{aligned}
&\big\langle \operatorname{ad}((X,a))^* G((Y,b)), (Z,c)\big\rangle = G((Y,b), \operatorname{ad}((X,a))(Z,c)) \\
&\quad = G\Big((Y,b), (-[X,Z], -\mathcal{L}_X c + \mathcal{L}_Z a)\Big) \\
&\quad = \int_{\mathbb{R}} Y'(x)\big(X'(x)Z(x) - X(x)Z'(x)\big)' + b(x)(-\mathcal{L}_X c(x) + \mathcal{L}_Z a(x))\,dx \\
&\quad = \int_{\mathbb{R}} Y'(x)\big(X''(x)Z(x) + X'(x)Z'(x) - X'(x)Z'(x) - X(x)Z''(x)\big)\,dx \\
&\qquad + \int_{\mathbb{R}} b(x)(-c'(x)X(x) + a'(x)Z(x))\,dx \\
&\quad = \int_{\mathbb{R}} Z(x)\big(X''(x)Y(x) + (X(x)Y'(x))'' + b(x)a'(x)\big)\,dx \\
&\qquad + \int_{\mathbb{R}} c(x)(b'(x)X(x) + b(x)X'(x))\,dx \\
&\quad = \Big\langle \big(X''Y + (XY')'' + ba', b'X + bX'\big), (Z,c)\Big\rangle
\end{aligned}
$$

Therefore the adjoint is given by

$$
\operatorname{ad}(X,a)^* G(Y,b)(x) = \big(X''Y + (XY')'' + ba', b'X + bX'\big)\ .
$$

Note that for general $(X,a),(Y,b) \in \mathcal{A}\times\mathcal{A}$ the adjoint is not an element of $G(\mathcal{A}\times\mathcal{A})$; the same statement is true for $\mathcal{A}_1 \times \mathcal{A}$. But for (X,a) equal (Y,b) we can rewrite the above equation similar as in 4.2 to obtain:

$$
\begin{aligned}
\operatorname{ad}(X,a)^* G(X,a)(x) &= \left(\frac{1}{2}(X'^2)' + \frac{1}{2}(X^2)''' + \frac{1}{2}(a^2)', a'X + aX'\right) \\
&= \left(\frac{1}{2}\left(\int_{-\infty}^x X'^2 + a^2 dx + (X^2)'\right)'', a'X + aX'\right) \\
&= \check{G}\left(\frac{1}{2}\left(\int_{-\infty}^x X'^2 + a^2 dx + (X^2)'\right), a'X + aX'\right)
\end{aligned}
$$

A similar argumentation as in Section 4.2 proves the existence of the adjoint and thus of the geodesic equation on $\widetilde{\mathcal{M}}$ and the non-existence on $\mathcal{M}$. $\square$

5.2. **Theorem** (The R-map for the 2HS equation). *Define the map*

$$
R : \begin{cases} \widetilde{M} \to \big(\mathcal{A}(\mathbb{R}, \mathbb{C}/\{-2\}), \|\cdot\|_{L^2}\big) \\ (\varphi,\alpha) \mapsto 2\,\varphi'^{1/2} e^{i\alpha/2} - 2\ . \end{cases}
$$

The R-map is invertible with inverse

$$
R^{-1} : \begin{cases} \mathcal{A}(\mathbb{R}, \mathbb{C}/\{-2\}) \to \widetilde{M} \\ \gamma \mapsto \Big(x + \frac{1}{4}\int_{-\infty}^x (|\gamma+2|^2 - 4)\,dx, 2\arg(\gamma(x)+2)\Big)\ . \end{cases}
$$

The pull-back of the flat L^2-metric via R is the metric G as defined in Theorem 5.1. Thus the space $\big(\widetilde{M}, G\big)$ is a flat space in the sense of Riemannian geometry.

Proof. A similar argument as in Section 4.3 shows that the image $R(\varphi,\alpha)$ is an element of $\mathcal{A}(\mathbb{R}, \mathbb{C}/\{-2\})$. The bijectivity follows form a straightforward calculation using that for $\gamma = R(\varphi,\alpha) = R(\mathrm{Id} + f, \alpha)$ we have

$$
\frac{1}{4}|\gamma(x) + 2|^2 - 1 = f'(x)\ ,
$$

which proves the identities $R \circ R^{-1} = R \circ R^{-1} = \mathrm{Id}$.

Since the mapping R is bijective, the pull-back via R yields a well-defined metric on $\widetilde{\mathcal{M}}$. To obtain its formula we have to calculate the tangent mapping of R. Let $(h,U) = (X \circ \varphi, U) \in T_{\varphi,\alpha}\widetilde{\mathcal{M}}$. We have:

$$T_{\varphi,\alpha}R(h,U) = \varphi_x^{-1/2} h' e^{i\alpha/2} + i\varphi_x^{1/2} e^{i\alpha/2} U \ .$$

Using this formula we have for $h = X_1 \circ \varphi, k = X_2 \circ \varphi$:

$$\begin{aligned} R^*\langle (h,U),(k,V)\rangle_{L^2} &= \mathrm{Re} \int_{\mathbb{R}} \langle T_{\varphi,\alpha}R(h,U), \overline{T_{\varphi,\alpha}R(k,V)}\rangle dx \\ &= \int_{\mathbb{R}} X_1'(x)X_2'(x) + \alpha(x)\beta(x) dx \\ &= G_{\varphi,\alpha}((h,U),(k,V)) \ . \end{aligned}$$

□

We can now again use this result to construct an analytic solution formula for the corresponding geodesic equation – the 2 component Hunter–Saxton equation.

5.3. **Theorem** (Solutions to the 2 HS–equation)**.** *Given an initial value* (u_0, ρ_0) *in* $\mathcal{A}_1(\mathbb{R}) \times \mathcal{A}(\mathbb{R})$ *the solution to the Hunter–Saxton equation is given by:*

$$(u,\rho) = (\varphi_t \circ \varphi^{-1}, -\alpha \circ \varphi + \alpha_t \circ \varphi^{-1}) \quad \textit{with} \quad (\varphi, \alpha) = R^{-1}(t(u_0' + i\rho_0)) \ .$$

In particular this means that a solution breaks in finite time T*, if and only if there exists a* $x \in \mathbb{R}$ *such that* $u_0'(x) < 0$ *and* $\rho_0(x) = 0$.

Proof. By the previous theorem we know that the path

$$(\varphi(t,x), \rho(t,x)) = R^{-1}(t\,\gamma_0)(x)$$

is a solution to the geodesic equation for every $\gamma_0 \in \mathcal{A}_1(\mathbb{R}, \mathbb{C}/\{-2\})$. It remains to choose γ such that the initial condition are satisfied. This can be achieved exactly by choosing $\gamma_0 = T_{\mathrm{Id},0}R(u_0,\rho_0) = (u_0' + i\rho_0)$. □

5.4. **Remark.** These theorems holds also in more general situations, i.e., for spaces $\mathcal{A}_1(\mathbb{R}) \times \mathcal{C}(\mathbb{R})$ with $\mathcal{A} \neq \mathcal{C}$, e.g., $H_1^\infty(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$. The result holds in this situation, since the diffeomorphism group $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ acts on $\mathcal{C}(\mathbb{R})$ for all choices of $\mathcal{A}$ and $\mathcal{C}$ among $C_c^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$ and $H^\infty(\mathbb{R})$.

6. Remarks on the periodic case

In this section we will briefly review the results of [26] and extend them to the case of real-analytic or ultra differentiable functions on the circle. In this section, $\mathrm{Diff}^\circ(S^1)/S^1$ denotes one of the following homogeneous spaces:

(1) $\mathrm{Diff}(S^1)/S^1$ the space of smooth diffeomorphisms on the circle modulo rotations.
(2) $\mathrm{Diff}^\omega(S^1)/S^1$ the space of real analytic diffeomorphisms on the circle modulo rotations, c.f. 2.7.
(3) $\mathrm{Diff}^{[M]}(S^1)/S^1$ the space of ultra differentiable diffeomorphisms of type Beurling or Roumieu on the circle modulo rotations, c.f. 2.8.

A diffeomorphism $\varphi \in \mathrm{Diff}^\circ(S^1)$ is related to its universal covering diffeomorphism $\widetilde{\varphi}$ by $\varphi(e^{ix}) = e^{i\widetilde{\varphi}(x)}$. Then $\widetilde{\varphi}(x) = x + f(x)$ where f is a 2π-periodic real valued function. Rotations correspond to constant functions f. Let $\widetilde{\mathrm{Diff}}^\circ(S^1)$ denote the

regular Lie group of lift to the universal cover of diffeomorphism. The corresponding homogeneous space is then $\widetilde{\mathrm{Diff}}^{\circ}(S^1)/\mathbb{R}$, factoring out all translations.

Theorem ([26]). *On the homogeneous space $\widetilde{\mathrm{Diff}}^{\circ}(S^1)/\mathbb{R}$ the square root representation is a bijective mapping given as follows,*

$$R : \begin{cases} \widetilde{\mathrm{Diff}}^{\circ}(S^1)/\mathbb{R} \to \big(\mathrm{Im}(R), \|\cdot\|_{L^2([0,2\pi])}\big) \subset \big(C^{\circ}_{2\pi\text{-}per}(\mathbb{R},\mathbb{R}_{>0}), \|\cdot\|_{L^2}\big) \\ \qquad\qquad \widetilde{\varphi} \mapsto 2\,(\widetilde{\varphi}')^{1/2} . \end{cases}$$

The image of the R-map is the sphere of radius $\sqrt{8\pi}$, i.e.,

$$\mathrm{Im}(R) = \Big\{ \gamma \in C^{\circ}_{2\pi\text{-}per}(\mathbb{R},\mathbb{R}_{>0}) : \|\gamma\|^2_{L^2} = \int_0^{2\pi} \gamma^2 \, d\theta = 8\pi \Big\} .$$

The pull-back of the restriction of the L^2-metric to $\mathrm{Im}(R)$ via R is the homogeneous Sobolev metric of order one, i.e.,

$$R^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\dot{H}^1} .$$

Thus the spaces $\big(\mathrm{Diff}^{\circ}(S^1)/S^1, \dot{H}^1\big)$ have constant positive sectional curvature.

Here $C^{\circ}_{2\pi-\mathrm{per}}(\mathbb{R},\mathbb{R}_{>0})$ denotes the corresponding space of sufficiently smooth functions, i.e., either the space of C^∞, real-analytic or ultra differentiable functions.

Proof. The case of C^∞-functions is proven in the work of Lenells [26]. The statement about the image of R follows from:

$$\|R(\widetilde{\varphi})\|^2 = 4\|\,(\widetilde{\varphi}')^{1/2}\|^2 = 4\int_0^{2\pi} \widetilde{\varphi}'(x)dx = 4\int_0^{2\pi} 1 + f'(x)dx = 8\pi$$

The remaining cases follow similarly using that $\mathrm{Diff}^\omega(S^1)$ and $\mathrm{Diff}^{[M]}(S^1)$ are Lie-subgroups of $\mathrm{Diff}(S^1)$, c.f. 2.7 and 2.8. □

As a direct consequence we get the following result:

6.1. **Theorem** (Solutions to the periodic HS-equation). *Given an initial value u_0 in $C^{\circ}(S^1,\mathbb{R}_{>0})$ the solution to the Hunter–Saxton equation stays locally in the same space. A solution exists for all time t, if and only if $u_0'(\theta) \geq 0$ for all $\theta \in S^1$.*

Remark. From our setup in 2.7 and 2.8 it is obvious that the results of [28] for the 2 component Hunter-Saxton equation extend to the cases of real-analytic and ultra differentiable functions.

7. A similar representation for the Camassa-Holm equation

In this article we have shown that certain "non-trivial" Riemannian spaces that have flat or constant curvature can be represented as a simple submanifold of the flat manifold of all sufficiently smooth functions equipped with the L^2-metric. In this section we will present a natural example of an metric space with "non-trivial" curvature that also can be represented as a (complicated) subspace of the flat manifold of all smooth functions, namely the Lie group $\mathrm{Diff}(S^1)$ equipped with the right invariant non-homogeneous H^1-metric.

Theorem ([17]). *The right invariant H^1-metric on the Lie group $\mathrm{Diff}(S^1)$ is given by*

$$G_\varphi(X\circ\varphi, Y\circ\varphi) = \int X(x)Y(x) + X'(x)Y'(x)\,dx ,$$

where X, Y are vector fields in the Lie algebra $\mathfrak{X}(S^1)$. The induced geodesic distance is positive and the corresponding geodesic equation is the Camassa-Holm equation [6]*, given by:*

$$\boxed{u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}\ .}$$

The geodesic equation is well-posed and the exponential map is a local diffeomorphism.

Using the ideas of [28] we can introduce a R-map for this space. We use again $\varphi(e^{ix}) = e^{i\widetilde{\varphi}(x)}$ with $\widetilde{\varphi}(x) = x + f(x)$ for periodic f. Let again $\widetilde{\mathrm{Diff}}(S^1)$ denote the regular Lie group of lifts to the universal cover of diffeomorphisms. For curves we get $\partial_t|_0 \varphi(t, e^{ix}) = i\partial_t|_0 \widetilde{\varphi}(t,x).e^{i\widetilde{\varphi}(x)} = i\partial_t|_0 f(t,x).e^{i\widetilde{\varphi}(x)}$. Thus, for tangent vectors we get $\delta\varphi = i.\delta\widetilde{\varphi}.\varphi = i.\delta f.\varphi$.

7.1. **Theorem.** *The R-map is defined by*

$$\begin{aligned} R(\widetilde{\varphi}) &:= 2\widetilde{\varphi}'^{\frac{1}{2}} e^{i(\widetilde{\varphi} - \mathrm{Id})/2} = 2(1+f')^{\frac{1}{2}} e^{if/2}, \\ R &: \widetilde{\mathrm{Diff}}(S^1) \to C^\infty_{2\pi\text{-}per}(\mathbb{R}, \mathbb{C})\ . \end{aligned}$$

The image under the R-map of the diffeomorphism group is the space $\mathcal{S}$ given by

$$\begin{aligned} \mathcal{S} := R(\widetilde{\mathrm{Diff}}(S^1)) &= \left\{ \gamma \in C^\infty_{2\pi\text{-}per}(\mathbb{R}, \mathbb{C} \setminus \{0\}) : F(\gamma) = (F_1(\gamma), F_2(\gamma)) = 0 \right\}, \\ \textit{where } F_1(\gamma) :&= \int_0^{2\pi} (|\gamma|^2 - 1)\, d\theta\ , \\ \textit{and } F_2(\gamma) :&= 8 \arg(\gamma)' - |\gamma|^2\ . \end{aligned}$$

The R map is injective and for any curve in $\mathcal{S}$ the inverse of R is given by:

$$R^{-1}(\gamma) = 2\arg(\gamma) + \mathrm{Id}_{\mathbb{R}} \qquad R^{-1} : \mathcal{S} \mapsto \mathrm{Diff}(S^1)\ .$$

Furthermore the pullback of the L^2-inner product on $C^\infty(S^1, \mathbb{C})$ to the diffeomorphisms group by the R-map is the right invariant Sobolev metric of order one.

Proof. To prove the characterization for the image of R we observe that for $\gamma \in C^\infty(S^1, \mathbb{C} \setminus \{0\})$ the function $R^{-1}(\gamma) \in C^\infty([0.2\pi), [0, 2\pi))$ is periodic if and only if $F_1(\gamma) = 0$. Furthermore we have that $R(R^{-1}(\gamma)) = \gamma$ if and only if $F_2(\gamma) = 0$.

To calculate the formula for the pullback metric we need to calculate the tangent of the R-map where h is tangent to $\widetilde{\varphi}$, i.e., to f.

$$T_{\widetilde{\varphi}} R h = \widetilde{\varphi}'^{-\frac{1}{2}} h' e^{if/2} + i\widetilde{\varphi}'^{\frac{1}{2}} e^{if/2} h\ ,$$

Thus the pullback of the L^2 inner product on $C^\infty(S^1, \mathbb{C})$ is given by:

$$\begin{aligned} (R^* \langle\ ,\ \rangle_{L^2})_{\widetilde{\varphi}}(h,h) &= \int_0^{2\pi} T_{\widetilde{\varphi}} R(h) \cdot \overline{T_{\widetilde{\varphi}} R(h)}\, dx \\ &= \int_0^{2\pi} \frac{h'^2}{\widetilde{\varphi}'} + h\widetilde{\varphi}'\, dx = \int_{S^1} X^2 + X'^2\, dx\ , \end{aligned}$$

with $h = X \circ \widetilde{\varphi}$. □

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